FROM MICROSCOPIC TRAFFIC MODELS TO FLUID TRAFFIC FLOW MODELS

M. RASCLE http://math.unice.fr/~rascle

Wien, May 05-09, 2008

OUTLINE.

- . MOTIVATION
- . 1.A: INTRODUCTION
- . 1.B: DISCRETE VIEW
- . 2 : LAGRANGIAN MACROSCOPIC VIEW
- . 3 : EULERIAN MACROSCOPIC VIEW
- . 4 : CASE WITH RELAXATION TERMS: ...
- . 5 : COMMENTS... MORE DETAILS... NOT IN THESE NOTES...

MOTIVATION

- Introduce a general class of 2^{nd} order models (AR)...

  in order to.... • be correct at small scales (~ 1 m, 1 sec): NO CRASH, NO negative velocities...

  • be (rigorously) based on microscopic models: SCALES

  • be very general and flexible: $\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t w + v \partial_x w = 0 \end{cases}$

  w: Lagrangian marker ("color"), (initially $w = v + p(\rho)$), with or without } influence on v,

  ex. destination...

  can be rigorously homogenized → with P. Bagnerini SIMA 2003 → with M. Herty, SIMA 2006

  preserves HJ approach as 1^{st} order models

- Prepare the discussion for round tables... Is this AR class too stable (at larger scales)? Answer: Yes! [How much] can we violate the subcharacteristic condition? Yes, but cleverly!...

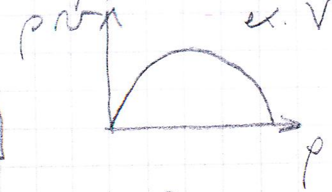
- Introduce a few mathematical ingredients for discussion:

  * Invariant regions: Chueh-Conley-Smoller
  * Subcharacteristic condition
  * BV estimates for Temple systems
  * CV of numerical schemes to weak (entropy) solutions ...

§1-A: Introduction: Essentially, 3 classes of models

- Microscopic models: ODE Follow the leader Models → ...
- Kinetic: not considered here
- Fluid:
  - 1st order Models: (LWR) $\partial_t \rho + \partial_x(\rho v) = 0$, $\boxed{v = V(\rho)}$ $V(\cdot) \searrow$ ex. $V(\rho) = 1 - \rho$

    equilibrium model



Very robust, (too) simple

[but associated Hamilton-Jacobi eq...]

  - 2nd order: 1) Payne-Whitham: cf Gas dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t v + v \partial_x v = -\frac{p'(\rho)}{\rho} \partial_x \rho \\ + \cancel{\text{Diffusion}} + \text{Relax} \end{cases}$$

...

... Daganzo (Requiem, 95)

PW is a terrible model!

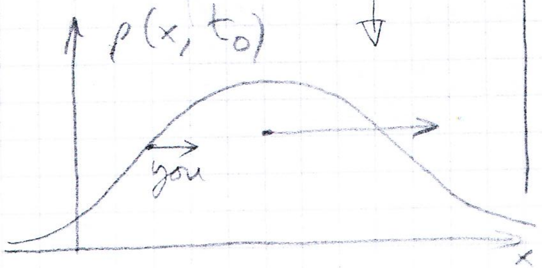
#1: $\exists$ cases where $v < 0$!!

#2: $\lambda_2 = v + c(\rho) > v$: some information travels faster than cars!!

2) Fixing: Aw-Rascle (Resurrection, SIAP 2000), Zhang (2002)

↳ In RHS, replace $\frac{\partial}{\partial x} \rightsquigarrow \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}$. Set $\tilde{p}'(\rho) := \frac{p'(\rho)}{\rho}$

Motivation:



$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t \underbrace{(v + p(\rho))}_{:= w} + v \partial_x \underbrace{(v + p(\rho))}_{w} = \begin{cases} 0 \\ \text{Relax} \to \dots \end{cases} \end{cases}$$

(Initial) (AR) $\longrightarrow$ $w = v + p(\rho)$

w := Lagrangian marker, {with / without} influence on v

...

§1.B. Discrete view:

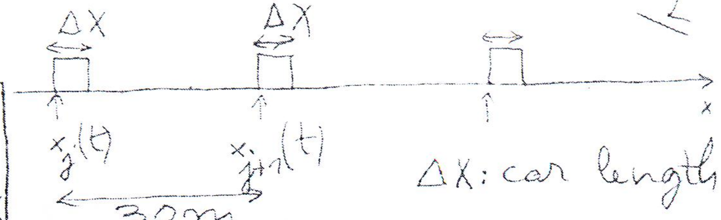
Gazis–Herman–Rothery type

$$\begin{cases}\dot{x}_j(t) = v_j \\ \dot{v}_j(t) = -P'\left(\dfrac{x_{j+1}(t)-x_j(t)}{\Delta X}\right)\cdot \dfrac{v_{j+1}(t)-v_j(t)}{\Delta X} \\ \qquad + \cancel{\dfrac{A}{T_r}}\left(V_{eq}\left(\dfrac{x_{j+1}(t)-x_j(t)}{\Delta X}\right) - v_j(t)\right)\end{cases}$$

(ODE)

convective term

relaxation term



Assume : $\boxed{A=0}$ ≠ Bando's model, cf subcharacteristic condition (Whitham)...

. Adimensional density / specific volume ... subtract j, $j+1$...

$$\rho_j \qquad \tau_j = \frac{x_{j+1}-x_j}{\Delta X} = \frac{1}{\rho_j} = s_j - 1,\ s_j = \left[x_{j+1} - (x_j + \Delta X)\right]\cdot\frac{1}{\Delta X}$$

= adimensional spacing

$$\begin{cases}\dot{\tau}_j(t) := \dot{\overline{\dfrac{x_{j+1}-x_j}{\Delta X}}}(t) \Rightarrow \\ \dot{v}_j(t) = -P'(\tau_j)\cdot\dot{\tau}_j(t) \Rightarrow\end{cases} \boxed{\begin{array}{l}\dot{\tau}_j(t) = \dfrac{v_{j+1}-v_j}{\Delta X} \\ \dot{w}_j(t) := \dot{\overline{(v_j + P(\tau_j))}}(t) = 0\end{array}}$$

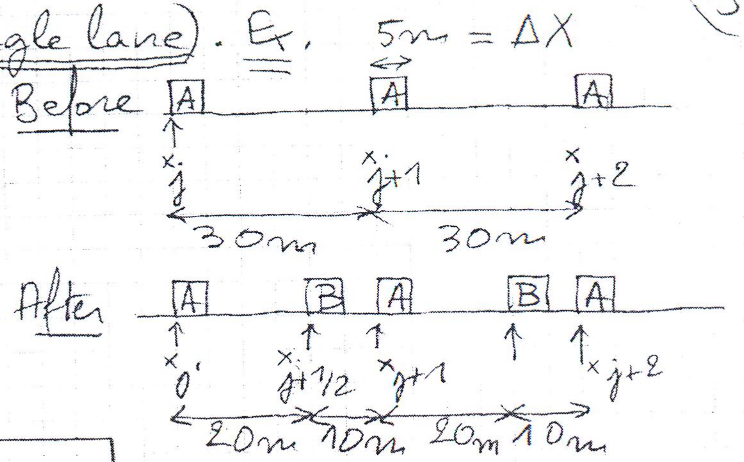
(ODE')

. Remark :

- Density $\rho = \frac{1}{\tau}$ is local, adimensional ⇒ unit independent ! $\rho \neq 1/\text{km}$!
- Later, hyperbolic scaling $(x,t) \rightsquigarrow (x',t') = (\varepsilon x, \varepsilon t)$, $X' = \varepsilon X$, $\Delta X' = \varepsilon \Delta X$

  ⇒ In this zoom, v, ρ, τ are scale independent
- $X_j := j\Delta X = \sum_{k=1}^{j} \frac{x_{k+1}(t) - x_k(t)}{\Delta X}\cdot \Delta X \rightsquigarrow \int^x \rho(y,t)\,dy := X(x,t)$ ...

. <u>Remark</u>: <u>τ is additive, not ρ</u> !! (<u>single lane</u>). <u>Ex.</u>



(old) $\tau_j = \frac{30}{5} = 6$

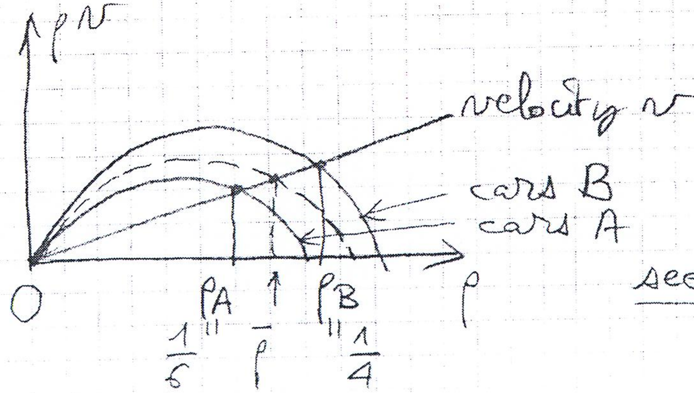
(new) $\tau_j := \frac{20}{5} = 4$

$\tau_{j+1/2} := \frac{10}{5} = 2$

$$\boxed{\begin{array}{l} 6 = 4 + 2 \\ 1/6 \neq 1/4 + 1/2 \ !! \end{array}}$$

Do it periodically, with same velocity v for all cars

<u>Homogenized flow</u>: average: $\bar{\tau} = \frac{1}{2}(4+2) = 3$. Equivalent averaged density: $\bar{\rho} := \frac{1}{\bar{\tau}} = \frac{1}{3}$ $\qquad \bar{\rho} \neq \overline{\left(\frac{1}{\tau}\right)}$!!



$\tau_A = \frac{1}{\rho_A}, \quad \tau_B = \frac{1}{\rho_B}, \quad \bar{\tau} = \frac{\tau_A + \tau_B}{2}$

<u>see</u> Aw, Klar, Materne, R SIAP, 02
Bagnerini, R SIMA, 03
Herty, R, SIMA, 06
" , Moutari, R, NHM 06

- Remark :
  - Simplest numerical approximation : 1rst order Euler explicit scheme

$$(\mathrm{ODE}')\ \boxed{\begin{cases} \dot{\tau}_j(t) = \dfrac{v_{j+1} - v_j}{\Delta X} \\ \dot{w}_j(t) = 0, \\ w_j(t) := v_j + P(\tau_j) \end{cases}} \ ; \ (\mathrm{FD}) \ \boxed{\begin{cases} \dfrac{\tau_j^{n+1} - \tau_j^n}{\Delta t} = \dfrac{v_{j+1}^n - v_j^n}{\Delta X} \\ w_j^{n+1} = w_j^n = \dots = w_j^0 := w_j \end{cases}}$$

ex. $P(\tau) = v_{max} - V_{eq}(\tau) \Rightarrow P'(\tau) = -V_{eq}'(\tau) < 0.$

- Original system is with delay. Delay vanishes in our scaling, (see below). Numerically, delay $\Longleftrightarrow$ explicit scheme !!!

- In the sequel, (FD) $\underset{\text{also}}{=}$ Godunov scheme for Lagrangian continuous system (L)

- Units system : 1 mn ; 1.5 km (# 1 mile) ; $v = \dfrac{1.5\,km}{mn} = 1$ $\Updownarrow$ 90 km/h
- Good mesh size :

$$\Delta X = 5m = \frac{1}{300} \times (1.5\,km) ; \ \Delta t = 0.2\ sec = \frac{1}{300}$$

quite reasonable for continuous model.

§2. Lagrangian macroscopic view:

- Hyperbolic scaling: $(x, X, t, \Delta X) \rightsquigarrow (x', X', t', \Delta X') := (\varepsilon x, \varepsilon X, \varepsilon t, \varepsilon \Delta X)$
- start from (FD), in rescaled coordinates. Then, at least formally,

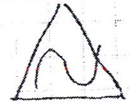
$$\frac{\tau_j^{n+1}-\tau_j^{n}}{\Delta t'} = \frac{v_{j+1}^{n}-v_j^{n}}{\Delta X'} \; ; \quad \frac{w_j^{n+1}-w_j^{n}}{\Delta t'} = 0$$

$$\downarrow$$

$$\frac{\partial \tau}{\partial t'}(X',t') = \frac{\partial v}{\partial X'}(X',t') \; ; \quad \frac{\partial w}{\partial t'}(X',t') = 0$$

when $\Delta X' = \varepsilon \Delta X \to 0$, $\Delta t' = \varepsilon \Delta t \to 0$, $\dfrac{\Delta t'}{\Delta X'} = C$ + CFL condition ...

with $t'_n = n\Delta t' = t$, $X'_j = j \Delta X' = X$

and corresponding initial data $U_0^{\varepsilon}(X') = U_0\left(\underset{\uparrow \text{fast}}{\frac{X'}{\varepsilon}}, \underset{\text{slow}}{X'}\right) := U^0(X, \varepsilon X)$ 

- or start from (ODE): $\dfrac{\partial \tau_j}{\partial t'} = \dfrac{v_{j+1}(t') - v_j(t')}{\Delta X}$; $\dfrac{d w_j}{d t'} = 0$: $\Delta t'$ is already 0

  ... then $\Delta X' \to 0$

- Limit system is: [with new $x, X, t \ldots := x', X', t'$] (drop the: $'$)

$$(L) \quad \boxed{\begin{cases} \partial_t \tau - \partial_X v = 0 \\ \partial_t w = 0 \end{cases} \quad w = v + P(\tau)}$$

Diagonalize: $\partial_t w = \partial_t v + P'(\tau)\underbrace{\partial_t \tau}_{= \partial_X v}$

$$\boxed{\begin{aligned} &\partial_t v + P'(\tau)\,\partial_X v = 0 \\ &\partial_t w = 0 \end{aligned}}$$

v, w: Riemann invariants.

6

. Eigenvalues: $\lambda_1 = P'(\tau) < 0 = \lambda_2$; λ_1 GNL genuinely nonlinear:

↳ shocks (braking) or rarefaction waves (acceleration)

λ_2: linearly degenerate (LD): contact discontinuities

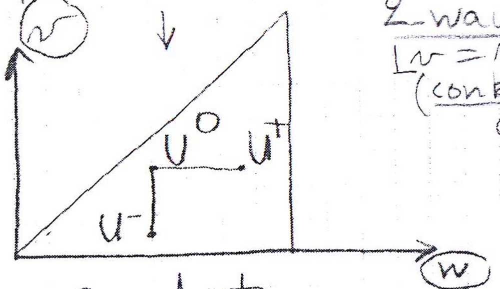
cars follow each other, at same speed

. Riemann Pb:

$$\boxed{\begin{cases} \partial_t \tau - \partial_X v = 0 \\ \partial_t w = 0 \end{cases} (L) \\ (\tau, w)(X,0) = (\tau_0(X), w_0(X)) := (\tau_\pm, w_\pm), \text{ for } \pm x > 0}$$

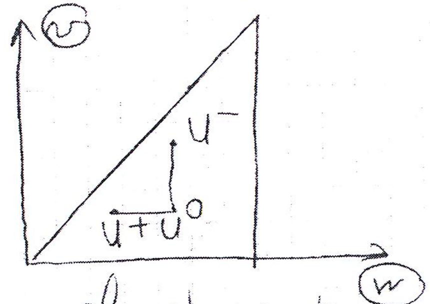
↳ 1-waves: $\boxed{w = w_-}$, even if shocks

2-waves: $v = v_+$ (contact) or



Rarefaction + contact, if $v_- < v_+$

acceleration



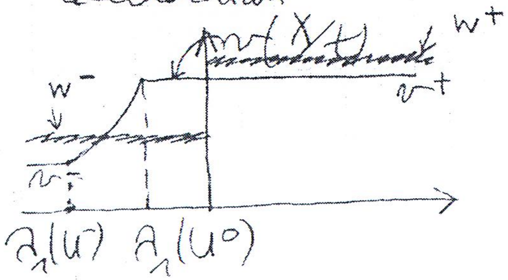
shock + contact if $v_- > v_+$

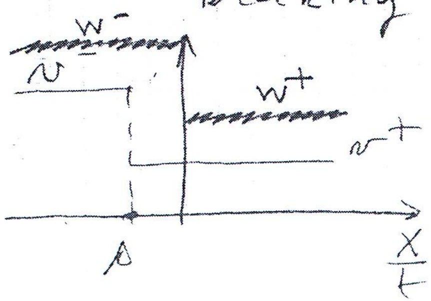
braking

THM: (Even if shock)

$$\boxed{\begin{array}{l} w(U^0) = w(U^-) : w^0 = w^- \\ \text{AND} \\ v(U^0) = v(U^+) : v^0 = v^+ \end{array}}$$

$\Rightarrow$ NO OSCILLATION IN v, w

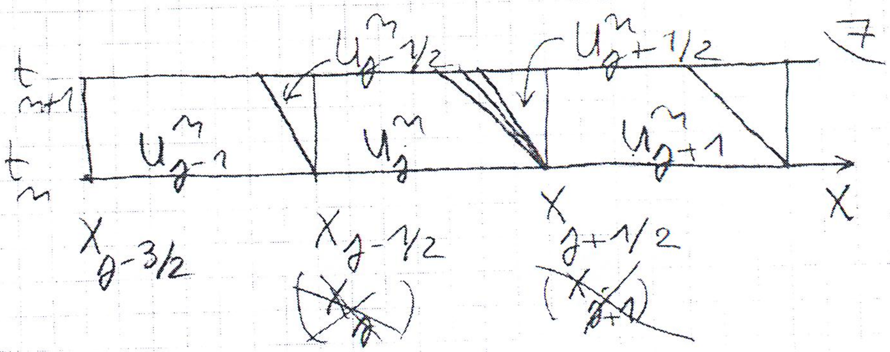




$$A = -\frac{P(\tau^0) - P(\tau^-)}{\tau^0 - \tau^-} = -\frac{v^0 - v^-}{\tau^0 - \tau^-}$$

Remark: compare with Payne-Whitham ↔ Gas dynamics !!!!

- <u>Lagrangian Godunov scheme</u>:



$$(L)\begin{cases}\partial_t \tau - \partial_X v = 0, \quad v = w - P(\tau)\\ \partial_t w = 0\end{cases}$$

$$U = \begin{pmatrix}\tau\\ w\end{pmatrix} \Longleftrightarrow \begin{pmatrix} v\\ w\end{pmatrix}$$

- At $t = t_n$, $U(X, t_n) \equiv U_j^n$ on $(X_{j-1/2}, X_{j+1/2})$

Integrate on $(X_{j-1/2}, X_{j+1/2}) \times (t_n, t_{n+1}) \Rightarrow \boxed{\underset{\text{top}}{\int}\tau - \underset{\text{bottom}}{\int}\tau + \underset{\text{right}}{\int} v - \underset{\text{left}}{\int} v = 0}$

$$\Rightarrow \boxed{0 = \Delta X \cdot \underbrace{\tau_j^{n+1}}_{\text{new value}} - \Delta X\, \tau_j^n + \Delta t\left(v^n_{j+1/2} - v^n_{j-1/2}\right)}$$

- Finite speed of propagation: If CFL condition satisfied, $\simeq$ as if Riemann Problem $U(X, t_n) = \begin{cases} U_j^n = U^-, & X < X_{j+1/2}\\ U_{j+1}^n = U^+, & X > \end{cases}$

$$\Rightarrow U^n_{j+1/2} = (w = w^-, v = v^+)$$

$$\Rightarrow \boxed{v^n_{j+1/2} = v^n_{j+1}}$$

- $\Rightarrow$ Formulas: (<u>God</u>) $\boxed{\dfrac{\tau_j^{n+1} - \tau_j^n}{\Delta t} = \dfrac{v^n_{j+1} - v^n_j}{\Delta X}; \quad w_j^{n+1} = \dots = w_j^0 = w_j}$ <u>Godunov</u>

(Godunov for (L) $\equiv$ FD for (ODE'))

- <u>THM</u>: Since w is constant on each cell, as for Riemann Pb, Total variation in X does not <u>increase in time</u> !!

. THM : . Moreover, $0 \leq v \leq v_{max}$, $0 \leq \rho = \frac{1}{\tau} \leq \rho_{max} = 1$.

(Compare with PW !)

. If $(\Delta X, \Delta t) \to 0$ with $\frac{\Delta t}{\Delta X} = C$ and CFL, (God) CV to (L)

Rigorous statement (!)

. If ΔX fixed and $\Delta t \to 0$, (God) $\equiv$ (FD) CV to (ODE') $\equiv$ (ODE)

AND (ODE') INHERITS THE BV ESTIMATES (NO OSCILLATION) FROM GODUNOV.

. Then when $\Delta t \to 0$, (ODE'), which is exactly the ½ discretization of (L), CV to the (unique) weak entropy solution of (L):

the two limits are the same: commutation of limits

. Finally, even for solutions with shocks,

Lagrangian system (L) $\Longleftrightarrow$ Eulerian system (E)

(away from vacuum)

§3 – Eulerian macroscopic view : Aw – R, SIAP, 2000; J. Greenberg, SIAP, 2002; M. Zhang, 2002

$$(E)\ \boxed{\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t w + v\,\partial_x w = 0 \end{cases}} \qquad w = v + p(\rho) = v + P(\tau) \qquad \tau = \frac{1}{\rho}$$

ex $p(\rho) = v_{max} - V_{eq}(\rho)$

- much more general, contains 1^{st} order models ($w \equiv C$),
  w = Lagrangian marker (color, affectation, trucks…)
- Lagrangian mass coordinates: Courant–Friedrichs

$$\begin{cases} \partial_t \rho = \partial_x(-\rho v) \ ; \\ \partial_t(\partial_x X) = \partial_x(\partial_t X) \end{cases} \qquad X = \int_{-\infty}^{x} \rho(y,t)\,dy = -\int_0^t \underbrace{(\rho v)}_{\text{cumulated flow}}(x,s)\,ds + f(x) + g(t)$$

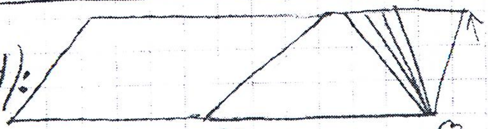
- $(x,t) \rightsquigarrow (X, T := t)$: $\dfrac{\partial}{\partial T} = \dfrac{\partial}{\partial t}\cdot 1 + \dfrac{\partial}{\partial x}\cdot \underbrace{v}_{\frac{\partial x}{\partial T}}$

$\dfrac{\partial}{\partial x} = 0 + \dfrac{\partial}{\partial X}\cdot \underbrace{\rho}_{\frac{\partial X}{\partial x}}$

- $\underbrace{\partial_t \rho + v\partial_x \rho}_{} + \rho\,\partial_x v = 0 = \dfrac{\partial \rho}{\partial T} + \rho\cdot\rho\,\dfrac{\partial v}{\partial X} \Rightarrow \dfrac{\partial}{\partial T}\left(-\dfrac{1}{\rho}\right) + \dfrac{\partial v}{\partial X} = 0$

$$(L)\ \boxed{\begin{cases} \dfrac{\partial \tau}{\partial T} - \dfrac{\partial v}{\partial X} = 0 \\ \dfrac{\partial w}{\partial T} = 0 \end{cases}}$$

- (E) $\Leftrightarrow$ (L), even for weak solutions
- (God):



$x^{n+1}_{j+1/2} = x^n_{j+1/2} + \Delta t\, v^n_{j+1}$

$x^n_{j-1/2}$ $\quad x^n_{j+1/2}$

- Link with Hamilton–Jacobi …

(10)

§4 - Case with relaxation term ... $\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, & w = v + p(\rho) \\ \partial_t(\rho w) + \partial_x(\rho w v) = \frac{A}{T_r}\rho(V_{eq}(\rho) - v), & \text{with} \end{cases}$

. If $A \neq 0$, Eulerian system becomes

$$V_{eq}(\rho) := \tilde{V}_{eq}\left(\frac{1}{\rho}\right)$$

. Whitham sub-characteristic condition: "convective term must dominate relaxation term"

. this condition is necessary for CV of the zero-relaxation limit. !!

. Here, in the scaling $(x,t) \rightsquigarrow (\varepsilon x, \varepsilon t) = (x', t')$, the relaxation term becomes $\frac{A}{\varepsilon T_r}\rho(V_{eq}(\rho) - v) \Rightarrow$ (Formally) CV to $v = V_{eq}(\rho)$

$$\partial_t \rho + \partial(\rho V_{eq}(\rho)) = 0 \quad \text{LWR}$$

(unless $A \rightsquigarrow \varepsilon A \Rightarrow \frac{A}{T_r}$ scale independent)

. Here, sub-charact. condition: $\boxed{-p'(\rho) \leqq V'_{eq}(\rho) \leq 0}$ cf J. Greenberg, AW: Ph-D, ...

$\boxed{\varepsilon \to 0}$ Chapman-Enskog expansion:

$v = V_{eq}(\rho) + \varepsilon u + \dots$

$\partial_t v + \partial_x(\rho V_{eq}(\rho)) = \varepsilon \partial_x(\underbrace{\rho V'_{eq}(\rho)(V'_{eq}(\rho) + p'(\rho))}_{\geq 0}\partial_x \rho) + \dots$

. Possible wild instabilities (Hadamard) if wrong sign. Formal arguments don't work in this case !! when $\varepsilon \to 0$:

. Example: CLL $\begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0 \\ \partial_t v^\varepsilon - \partial_x w^\varepsilon = 0 \\ \partial_t w^\varepsilon = \frac{1}{\varepsilon}(u^\varepsilon - w^\varepsilon) \end{cases}$ Formal limit: $\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v - \partial_x u = 0 \end{cases}$ $\Leftrightarrow \Delta_{x,t}\varphi = 0$!!!

. Now, for fixed ε .. see oral discussion. □