Almost Diagonalization of Pseudodifferential Operators with Respect to Coherent States (Gabor Frames)

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Outline

Aspects

- Gabor frames = discretized (generalized) coherent states
- convenient for interpretation in physics and signal processing — contribution of cells in phase-space
- new results on classical pseudodifferential operators
- applications in wireless communication
- computational physics?

Pseudodifferential Operators

Symbol σ on phase space $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$

$$\sigma(x, D)f(x) = \int_{\mathbb{R}^{2d}} \sigma(x, \xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi$$

Hörmander classes $S^m_{\delta,\rho}$ as standard symbol classes for PDE

In phase-space analysis

$$\sigma \in S^0_{0,0} \;\Leftrightarrow\; \partial^\alpha \sigma \in L^\infty(\mathbb{R}^{2d}), \quad \forall \alpha \geq 0$$

Standard Results

Boundedness.

Theorem (Calderòn-Vaillancourt)

If $\sigma \in S^0_{0,0}$, then $\sigma(x,D)$ is bounded on $L^2(\mathbb{R}^d)$ and

$$\|\sigma(x,D)\|_{L^2\to L^2} \leq \sum_{|\alpha|\leq 2d+1} \|\partial^\alpha \sigma\|_\infty .$$

Functional Calculus.

Theorem (Beals '77)

If $\sigma \in S^0_{0,0}$ and $\sigma(x,D)$ is invertible on $L^2(\mathbb{R}^d)$, then $\sigma(x,D)^{-1} = \tau(x,D)$ for some $\tau \in S^0_{0,0}$.

REMARK: NO asymptotic expansions, NO symbolic calculus for $S^0_{0,0}$.

Phase-Space Shifts, Coherent States

Phase-space shifts: $z = (x, \xi) \in \mathbb{R}^{2d}$, $f \in L^2(\mathbb{R}^d)$.

$$\pi(z)f(t) = e^{2\pi i \xi \cdot t} f(t - x) = M_\xi T_x f(t)$$

$\{\pi(z)g : z \in \mathbb{R}^{2d}\}$ is a set of (generalized) coherent states. Continuous resolution of identity (phase-space decomposition):

$$f = \langle \gamma, g \rangle^{-1} \int_{\mathbb{R}^{2d}} \langle f, \pi(z)g \rangle \, \pi(z)\gamma \, dz$$

Often $g(t) = g(t) = e^{-\pi t^2}$ Gaussian

Short-time Fourier transform (cross Wigner distribution, Gabor transform, radar ambiguity function, coherent state transform, etc.) of f with respect to state/window g

$$V_g f(z) = \langle f, \pi(z)g \rangle = \Big(f \cdot g(\cdot - x) \Big)\widehat{\ }(\xi)$$

measures “amplitude” of f in neighborhood of point z in phase-space (local frequency amplitude ξ near time x)

Discrete Expansions

Discretize the continuous resolution of the identity

- g "nice", e.g., $g \in \mathcal{S}$
- $\Lambda \subseteq \mathbb{R}^{2d}$ lattice, $\Lambda = A\mathbb{Z}^{2d}$ for $A \in GL(2d, \mathbb{R})$, e.g., $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$.

Wanted: stable expansions

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma\rangle \, \pi(\lambda)g \tag{1}$$

for suitable pair of "nice" g, γ with unconditional convergence and equivalence of norms on f and norm on the coefficients.

Gabor Frames

(1) is equivalent to the following:

- $\{\pi(\lambda)g, \lambda \in \Lambda\}$ is a frame (Gabor frame), i.e., $\exists A, B > 0$, such that

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d)\,.$$

If $A = B$, then $\{\pi(\lambda)g, \lambda \in \Lambda\}$ is called a tight frame and

$$f = A^{-1}\sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)g\rangle\, \pi(\lambda)g$$

- looks like orthonormal expansion
- but $\{\pi(\lambda)g : \lambda \in \Lambda\}$ is no basis, coefficients not unique
- Smoothness w.r.t. phase-space content — modulation spaces — results on nonlinear approximation

The Sjöstrand Class

$$\|\sigma\|_{M^{\infty,1}} = \int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |(V_\Phi\sigma(z,\zeta)|\, d\zeta < \infty$$

$\zeta \to V_\Phi\sigma(z,\zeta) = \quad (\sigma \cdot T_z\Phi)\widehat{\ } \in L^1$.
$\Rightarrow$ σ is bounded and locally in $\mathcal{F}L^1$!
$M^{\infty,1}$ contains functions without smoothness.

Weighted Sjöstrand class $M_v^{\infty,1}(\mathbb{R}^{2d})$.

$$\|\sigma\|_{M_v^{\infty,1}} = \int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |(\sigma\cdot T_z\Phi)\widehat{\ }(\zeta)|\, v(\zeta)\, d\zeta < \infty$$

$M_v^{\infty,\infty}$ with norm

$$\|\sigma\|_{M_v^{\infty,\infty}} = \sup_{z,\zeta\in\mathbb{R}^{2d}} |(\sigma\cdot T_z\Phi)\widehat{\ }(\zeta)| v(\zeta)$$

Observation: If $v_s(\zeta) = (1+|\zeta|)^s$, then

$$S^0_{0,0} = \bigcap_{s\geq 0} M^{\infty,1}_{v_s} = \bigcap_{s\geq 0} M^{\infty,\infty}_{v_s}$$

Matrix of $\sigma(x, D)$ with respect to Gabor Frame

Natural idea: investigate pseudodifferential operators with respect to coherent states/phase-space shifts (quantum mechanics, quantum optics?)
Assume that $\{\pi(\lambda)g : \lambda \in \Lambda\}$ is a (tight) frame for $L^2(\mathbb{R}^d)$. Then
$f = \sum_{\lambda\in\Lambda}\langle f, \pi(\mu)g\rangle\pi(\mu)g$ and
$\sigma(x, D)(\pi(\mu)g) = \sum_{\lambda\in\Lambda}\langle\sigma(x, D)\pi(\mu)g, \pi(\lambda)g\rangle\pi(\lambda)g$.

$$\begin{aligned}
\sigma(x, D)f &= \sum_{\mu\in\Lambda}\langle f, \pi(\mu)g\rangle\sigma(x, D)\pi(\mu)g \\
&= \sum_{\lambda\in\Lambda}\Big(\sum_{\mu\in\Lambda}\underbrace{\langle\sigma(x, D)\pi(\mu)g, \pi(\lambda)g\rangle}\langle f, \pi(\mu)g\rangle\Big)\pi(\lambda)g
\end{aligned}$$

Stiffness Matrix

Matrix of $\sigma(x,D)$ is $M(\sigma)_{\lambda,\mu} = \langle \sigma(x,D)\pi(w)g, \pi(z)g\rangle$
Stiffness matrix, channel matrix

$$
\begin{array}{ccc}
L^2(\mathbb{R}^d) & \stackrel{\sigma(x,D)}{\longrightarrow} & L^2(\mathbb{R}^d) \\
\downarrow V_g|_\Lambda & & \downarrow V_g|_\Lambda \\
\ell^2(\Lambda) & \stackrel{M(\sigma)}{\longrightarrow} & \ell^2(\Lambda)
\end{array}
\tag{2}
$$

$$
\begin{aligned}
\langle \sigma(x,D)\pi(w)g, \pi(z)g\rangle &= \langle \sigma, R(\pi(z)g, \pi(w)g)\rangle \\
&= \langle \sigma, M_{\zeta(z,w)}T_{u(z,w)}R(g,g)\rangle = V_\Phi\sigma(u,\zeta)
\end{aligned}
$$

- $R(f,g)(x,\xi) = f(x)\overline{\hat{g}(\xi)}e^{-2\pi i x\cdot\xi}$ Rihaczek distribution
- phase-space properties of σ $\Leftrightarrow$ off-diagonal decay of $M(\sigma)$

Almost Diagonalization for the Sjöstrand Class I

Theorem

Fix $g \neq 0$, such that $\int_{\mathbb{R}^{2d}} |V_g g(z)| v(z)\, dz < \infty$ ($g \in M^1_v$)
(A) A symbol $\sigma \in M^{\infty,1}_v$, if and only if there is $H \in L^1_v(\mathbb{R}^{2d})$, such that

$$|\langle \sigma(x,D)\pi(w)g, \pi(z)g\rangle| \leq H(z-w) \qquad w,z \in \mathbb{R}^{2d} \tag{3}$$

(B) Assume in addition that $\{\pi(\lambda)g : \lambda \in \Lambda\}$ is a tight frame. Then $\sigma \in M^{\infty,1}_v$, if and only if there is $h \in \ell^1_v(\Lambda)$, such that

$$|\langle \sigma(x,D)\pi(\mu)g, \pi(\lambda)g\rangle| \leq h(\lambda-\mu) \qquad \lambda,\mu \in \Lambda\,. \tag{4}$$

- Matrix of $\sigma(x,D)$ is dominated by convolution kernel in ℓ^1_v.
- If $v(x+y) \leq v(x)v(y)$, then ℓ^1_v is Banach algebra w.r.t. convolution.

Consequence: if $\sigma_1, \sigma_2 \in M^{\infty,1}_v$, then

$$\sigma_1(x,D)\sigma_2(x,D) = \tau(x,D) \qquad \text{for } \tau \in M^{\infty,1}_v$$

Almost Diagonalization II

Theorem

Fix $g \neq 0$, such that $\int_{\mathbb{R}^{2d}} |V_g g(z)| v(z)\, dz < \infty$ ($g \in M^1_v$). Assume that $v^{-1} * v^{-1} \leq C v^{-1}$.

(A) A symbol $\sigma \in M_v^{\infty,\infty}$, if and only if

$$|\langle \sigma(x,D)\pi(w)g), \pi(z)g\rangle| \leq C\, v(z-w)^{-1} \qquad w,z \in \mathbb{R}^{2d} \tag{5}$$

(B) Assume in addition that $\{\pi(\lambda)g : \lambda \in \Lambda\}$ is a tight frame. Then $\sigma \in M_v^{\infty,1}$, if and only if

$$|\langle \sigma(x,D)\pi(\mu)g), \pi(\lambda)g\rangle| \leq C' v(\lambda-\mu)^{-1} \qquad \lambda,\mu \in \Lambda\,. \tag{6}$$

Stiffness matrix possesses quantifiable off-diagonal decay.

Almost Diagonalization for Hörmander Class

Corollary

Fix $g \in \mathcal{S}$ *and tight Gabor frame* $\{\pi(\lambda)g : \lambda \in \Lambda\}$. *TFAE:*

(A) $\sigma \in S^0_{0,0}$

(B) $|\langle \sigma(x, D)\pi(w)g), \pi(z)g\rangle| = \mathcal{O}\left(|z - w|^{-N}\right)$ *for all* $N \geq 0$.

(C) $|\langle \sigma(x, D)\pi(\mu)g), \pi(\lambda)g\rangle| = \mathcal{O}\left(|\lambda - \mu|^{-N}\right)$ *for all* $N \geq 0$.

Stiffness matrix of symbol in $S^0_{0,0}$ decays rapidly off diagonal.

$M_v^{\infty,1}$ is Inverse-Closed

Theorem (Sjöstrand)

If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and $\sigma(x,D)$ is invertible on $L^2(\mathbb{R}^d)$, then $\sigma(x,D)^{-1} = \tau(x,D)$ for some $\tau \in M^{\infty,1}$.

Theorem

Assume that v is submultiplicative and

$$\lim_{n\to\infty} v(nz)^{1/n} = 1, \quad \forall z \in \mathbb{R}^{2d}.$$

If $\sigma \in M_v^{\infty,1}(\mathbb{R}^{2d})$ and $\sigma(x,D)$ is invertible on $L^2(\mathbb{R}^d)$, then $\sigma(x,D)^{-1} = \tau(x,D)$ for some $\tau \in M_v^{\infty,1}$.

- Only functional calculus, neither symbolic calculus nor asymptotic expansions
- Even if $\sigma(x,D)$ is invertible on $L^2(\mathbb{R}^d)$, $M(\sigma)$ is not invertible on $\ell^2(\Lambda)$, but it possess a pseudoinverse with same off-diagonal decay as $M(\sigma)$.

Approximation by Elementary Operators

Stiffness matrix possesses strong off-diagonal decay, i.e., can be approximated well by banded matrix.
Definition: Gabor multipliers If $\{\pi(\lambda)g, \lambda \in \Lambda\}$ is a tight frame and $\mathbf{a} \in \ell^\infty(\mathbb{Z}^{2d})$, define

$$\mathcal{M}_{\mathbf{a}} f = \sum_{\lambda \in \Lambda} a_\lambda \langle f, \pi(\lambda)g\rangle \pi(\lambda)g$$

[if $a_\lambda = 1$, then $M_{\mathbf{a}} = \mathrm{Id}$.]
Diagonal of $M(\sigma)$ corresponds to the operator

$$\mathcal{M}_d f = \sum_{\lambda \in \Lambda} \underbrace{\langle \sigma(x, D)\pi(\lambda)g, \pi(\lambda)g\rangle} \langle f, \pi(\lambda)g\rangle \, \pi(\lambda)g$$

Approximation by Elementary Operators II

Side-diagonals correspond to operators of the form

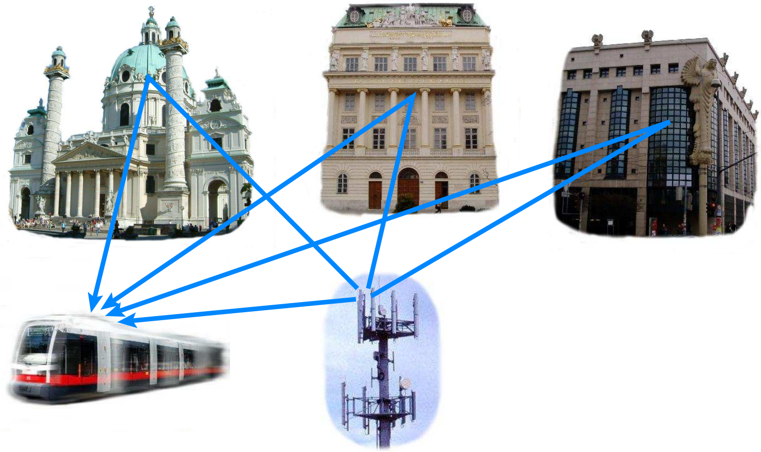
$$\begin{aligned}\mathcal{M}f &= \sum_{\lambda\in\Lambda} \underbrace{\langle \sigma(x,D)\pi(\lambda)g, \pi(\lambda-\kappa)g\rangle}\, \langle f, \pi(\lambda)g\rangle\, \pi(\lambda-\kappa)g \\ &= \pi(-\kappa) \sum_{\lambda\in\Lambda} b_\lambda \langle f, \pi(\lambda)g\rangle\, \pi(\lambda)g\end{aligned}$$

Approximation of $M(\sigma)$ by banded matrix amounts to approximation of $\sigma(x,D)$ by modified Gabor multipliers

$$\sigma(x,D)f \approx \sum_{|\kappa|\leq L} \pi(-\kappa)\mathcal{M}_{\mathbf{a}_\kappa} f$$

(Error estimates: Andreas Klotz, KG, 200?)

Time-Varying Systems



Time-Varying Channels

Received signal $\tilde{f}$ is a superposition of time lags

$$\tilde{f}(t) = \int_{\mathbb{R}^d} V(u) \dots f(t+u)\, du$$

Received signal $\tilde{f}$ is a superposition of frequency shifts

$$\tilde{f}(t) = \int_{\mathbb{R}^d} W(\eta) \dots e^{2\pi i \eta t} f(t) d\eta$$

Thus received signal $\tilde{f}$ is a superposition of time-frequency shifts:

$$\widetilde{f}(t) \quad = \quad \int_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta, u) \underbrace{e^{2\pi i \eta \cdot t} f(t+u)}_{(\pi(-u,\eta)f)(t)} \, du d\eta$$

Modelling

Standard assumption of engineers: $\sigma \in L^2$ and $\hat{\sigma}$ has compact support.

Problem: Does not include distortion free channel and time-invariant channel.

So supp $\hat{\sigma}$ is compact, but $\hat{\sigma}$ is “nice” distribution. Then σ is bounded and an entire function.

$\boxed{\Rightarrow \ \sigma \in M_v^{\infty,1} \text{ for exponential weight.}}$

Multiplexing

Transmission of “digital word” $(c_k), c_k \in \mathbb{C}$ via pulse g

$$f(t) = \sum_{k=0}^{\infty} c_k g(t-k)$$

Transmission of several “words” ($\Longleftrightarrow$ simultaneous transmission of a symbol group) by distribution to different frequency bands with modulation

Partial signal for ℓ-th word $\mathbf{c}^{(\ell)} = (c_{kl})_{k\in\mathbb{Z}}$

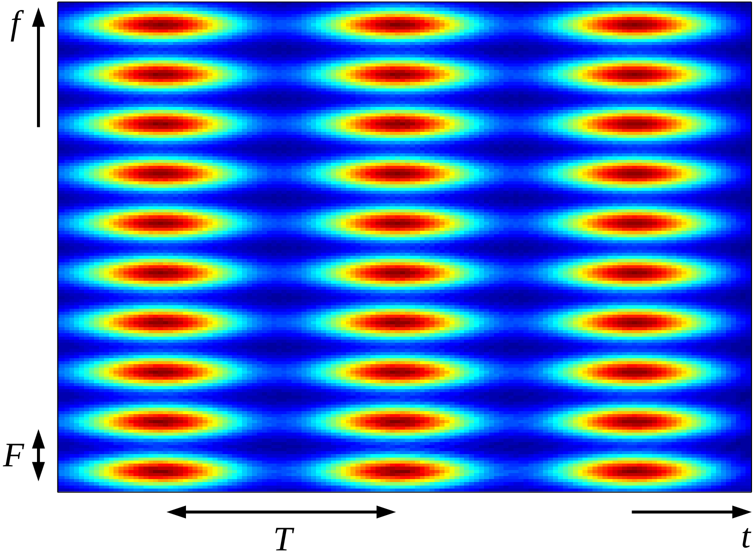
$$f_\ell = \sum_k c_{kl} T_k g$$

Total signal is a Gabor series (Gabor expansion)

$$f_\ell = M_{\theta\ell}\Big(\sum_k c_{kl} T_k g\Big)$$

$$f = \sum_{k,l} c_{kl} M_{\theta l} T_k g = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g$$

If $M_\theta T_k g$ orthogonal, then OFDM (orthogonal frequency division


f
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Decoding and the Channel Matrix

Received signal is

$$\widetilde{f} = \sigma(x, D)\Big(\sum_{\mu \in \Lambda} c_\mu \pi(\mu) g\Big)$$

Standard procedure: take correlations

$$\langle \widetilde{f}, \pi(\lambda)g\rangle = \sum_{\mu} c_\mu \, \langle \sigma(x, D)\pi(\mu)g, \pi(\lambda)g\rangle$$

Solve the system of equations

$$\mathbf{y} = A\mathbf{c}$$

where $A_{\lambda,\mu} = \langle \sigma(x, D)(\pi(\mu)g), \pi(\lambda)g\rangle$ is the channel matrix.

Decoding II

Recovery of original information c_λ amounts to inversion of channel matrix (equalization, demodulation).

Engineer's assumption in statistical models: A is a diagonal matrix i.e.,

$$c_\lambda = \langle \sigma(x, D)\pi(\lambda)g, \pi(\lambda)g\rangle^{-1} y_\lambda$$

Cannot quite be true, but A is almost diagonal.
Hope: improvement of accurary by including side-diagonal.

Final remarks

- Use the almost diagonalization w.r.t. Gabor frames in wireless communications and in quantum mechanics
- Approximation by banded matrices is simple.
- Works only on $\mathbb{R}^d$, not on domains
- Any advantages from adaptive methods (CDD1 and CDD2)? [Dahlke, Fornasier, KG]