Semi-Lagrangian adaptive schemes for the Vlasov equation

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joint work Albert Cohen (Paris 6), Michel Mehrenberger and Eric Sonnendrücker (Strasbourg)

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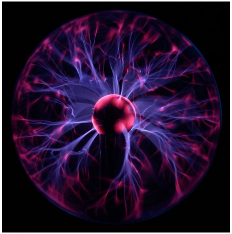
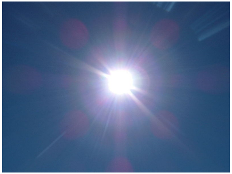
Outline

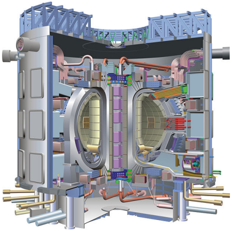
Outline

Introduction

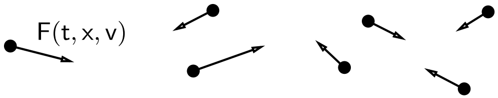




- Plasma: gas of charged particles (as in stars or lightnings)
- Applications: controlled fusion, Plane/flame interaction...



Models for plasma simulation



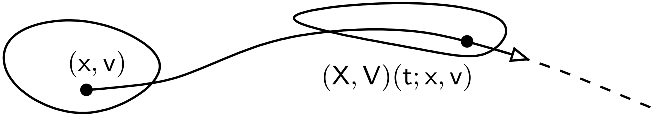
- Microscopic model $\rightsquigarrow$ N body problem in 6D phase space
- Kinetic models: statistical approach, replace particles $\{x_i(t), v_i(t)\}_{i \leq N}$ by a distribution density $f(t, x, v)$
  - binary collisions $\rightsquigarrow$ Bolztmann equation
  - mean-field approximation $\rightsquigarrow$ Vlasov equation

$$\partial_t f(t, x, v) + v\ \partial_x f(t, x, v) + F(t, x, v)\ \partial_v f(t, x, v) = 0$$

- Fluid models: assume f is maxwellian and compute only first moments: density $n(t, x) := \int f \,\mathrm{d}v$, momentum $u(t, x) := n^{-1} \int vf \,\mathrm{d}v$ and pressure $p := \int f(v-u)^2 \,\mathrm{d}v$.

Vlasov equation as a "smooth" transport equation

- Existence of smooth solutions (cf. Iordanskii, Ukai-Okabe, Horst, Wollman, Bardos-Degond, Raviart...)
- density f is constant along characteristic curves,

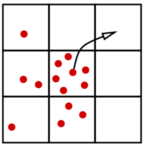


- Characteristic flow is a measure preserving diffeomorphism

$$\mathcal{F}(t) \ : \ (x, v) \rightarrow (X, V)(t; x, v)$$

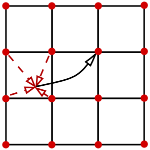
$$\mathcal{B}(t) \ : \ (X, V)(t; x, v) \rightarrow (x, v)$$

Numerical methods for the Vlasov equation



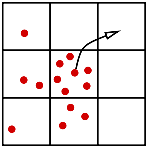
$\{(x_i(t), v_i(t)) : i \leq N\}$

$\{f_i(t) : i \leq N\}$



- Particle-In-Cell (PIC) methods ([Harlow 1955])
  - Hockney-Eastwood 1988, Birdsall-Langdon 1991 (physics)
  - Neunzert-Wick 1979, Cottet-Raviart 1984, Victory-Allen 1991, Cohen-Perthame 2000 (mathematical analysis)
- Eulerian (grid-based) methods
  - Forward semi-Lagrangian [Denavit 1972]
  - Backward semi-Lagrangian [Cheng-Knorr 1976, Sonnendrücker-Roche-Bertrand-Ghizzo 1998]
  - Conservative flux based methods [Boris-Book 1976, Fijalkow 1999, Filbet-Sonnendrücker-Bertrand 2001]
  - Energy conserving FD Method: [Filbet-Sonnendrücker 2003]

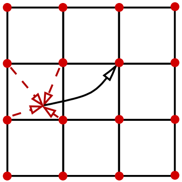
the Particle-In-Cell method



- Principle: approach the density distribution f by transporting sampled "macro-particles"
  - initialization: deterministic approximation of f_0 $\rightsquigarrow$ macro-particles $\{x_i(0), v_i(0)\}_{i \leq N}$
  - knowing the charge and current density, solve the Maxwell system
  - knowing the EM field, transport the macro-particles along characteristics
- Benefits: intuitive, good for large & high dimensional domains
- Drawback: sampling in general performed by Monte Carlo $\rightsquigarrow$ poor accuracy

the (backward) semi-Lagrangian method

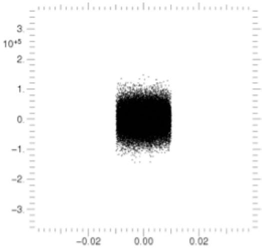
$\{f_i(t) : i \leq N\}$



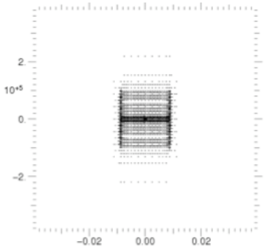
- Principle: use a transport-interpolation scheme
  - initialization: projection of f_0 on a given FE space
  - knowing f, compute the charge and current densities and solve the Maxwell system
  - Knowing the EM field, transport and interpolate the density along the flow.
- Benefits: good accuracy, high order interpolations are possible
- Drawback: needs huge resources in 2 or 3D

Comparison

- Initializations of a semi-gaussian beam in 1+1 d



PIC code



non linear approximation

- Solution: use an adaptive semi-Lagrangian scheme !

Outline

1. Mathematical modeling of charged particles
   - Applications and models
   - The Vlasov equation
   - Numerical methods

2. The adaptive semi-Lagrangian approach
   - Notations
   - Error analysis
   - The prediction-correction scheme
   - Numerical results

Adaptive semi-Lagrangian scheme: notations

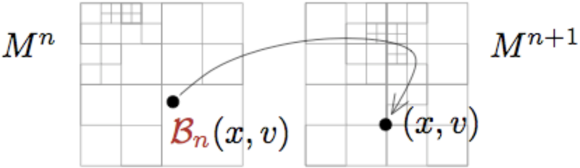
- Knowing $f_n \approx f(t_n := n\Delta t)$, approach the backward flow

$$\mathcal{B}(t_n) : (x, v) \to (X, V)(t_n; t_{n+1}, x, v)$$

by a diffeomorphism $\mathcal{B}_n = \mathcal{B}[f_n]$

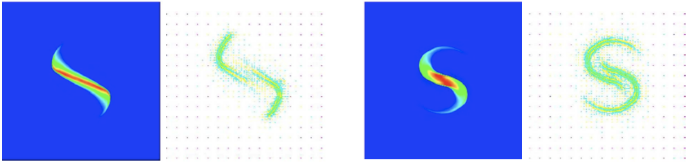
- transport the numerical solution with $\mathcal{T} : f_n \to f_n \circ \mathcal{B}_n$
- then interpolate on the new mesh M^{n+1}:

$$f_{n+1} := P_{M^{n+1}} \mathcal{T} f_n$$



Prior schemes

Gutnic, Haefele, Paun, Sonnendrücker *Comput. Phys. Comm. 2004*



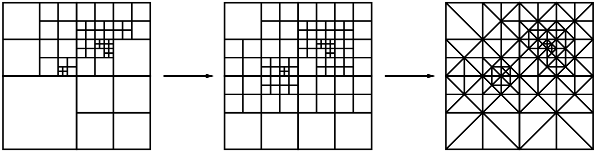
- Use interpolets on multilevel octrees
- Hierarchical grid is transported by advecting the nodes forward in time and creating cells of same level in new grid
- Related work on adaptive Lagrange-Galerkin methods for unsteady convection-diffusion problems

  Houston, Süli *Technical report 1995, Math. Comp. 2001*

A second approach

CP, Mehrenberger *Proceedings of Cemracs 2003*

- hierarchical conforming $\mathcal{P}^1$ FE spaces build on quad meshes



- the corresponding interpolation P_M satisfies

$$\|(I - P_M)f\|_{L^\infty} \lesssim \sup_{\alpha \in M} |f|_{W^{2,1}(\alpha)}$$

- for given f , construct $M := \mathbb{A}_\varepsilon(f)$ by adaptive splittings

Analysis of the uniform scheme

Besse *SINUM 2004*

- Error: decompose $e_{n+1} := \|f(t_{n+1}) - f_{n+1}\|_{L^\infty}$ into

$$e_{n+1} \le \|f(t_{n+1}) - \mathcal{T} f(t_n)\|_{L^\infty} + \|\mathcal{T} f(t_n) - \mathcal{T} f_n\|_{L^\infty} + \|(I - P_{\mathcal{K}})\mathcal{T} f_n\|_{L^\infty},$$

and using a 2nd order time splitting scheme for $\mathcal{T}$, show

$$e_{n+1} \le (1 + C(T)\Delta t)e_n + C(T)(\Delta t^3 + h^2), \quad n\Delta t \le T$$

as long as $f_0 \in W^{2,\infty}(\mathbb{R}^2)$. Hence $e_n \le C(T)(\Delta t^2 + h^2/\Delta t)$.

- Complexity: balance with $\Delta t^2 \sim h^2/\Delta t$, so that

$$\boxed{e_n \le C(T)h^{4/3} \le C(T)N_h^{-2/3}} \qquad (N_h \sim h^{-2})$$

Analysis of the adaptive scheme

- Decompose again $e_{n+1} := \|f(t_{n+1}) - f_{n+1}\|_{L^\infty}$ into

$$e_{n+1} \leq \|f(t_{n+1}) - \mathcal{T} f(t_n)\|_{L^\infty} + \|\mathcal{T} f(t_n) - \mathcal{T} f_n\|_{L^\infty} + \|(I - P_{M^{n+1}})\mathcal{T} f_n\|_{L^\infty},$$

and estimate

$$e_{n+1} \leq (1 + C(T)\Delta t\,)e_n + C(T)\Delta t^{\,3} + \|(I - P_{M^{n+1}})\mathcal{T} f_n\|_{L^\infty}$$

as long as $f_0 \in W^{1,\infty}(\mathbb{R}^2)$.

- $\rightsquigarrow$ goal: predict M^{n+1} such that it is ε-adapted to $\mathcal{T} f_n$, ie

$$\sup_{\alpha \in M^{n+1}} |\mathcal{T} f_n|_{W^{2,1}(\alpha)} \leq \varepsilon$$

Adaptive mesh prediction, I

- Goal: given M^n and f_n, build M^{n+1} in such a way that

$$\sup_{\alpha \in M^{n+1}} |\mathcal{T} f_n|_{W^{2,1}(\alpha)} \leq \varepsilon$$

- Idea: use adaptive splitting.
- $\rightsquigarrow$ Questions:
  - Q_1: which cells should be refined in M^{n+1} ?
  - Q_2: how big can $|\mathcal{T} f_n|_{W^{2,1}(\alpha)} = |f_n \circ \mathcal{B}_n|_{W^{2,1}(\alpha)}$ be ?
  - Q_3: is $\mathcal{T}$ stable with respect to the curvature, ie

$$|\mathcal{T} f_n|_{W^{2,1}(\alpha)} \leq C |f_n|_{W^{2,1}(\mathcal{B}_n(\alpha))} \quad ?$$

Adaptive mesh prediction, II

- Q_3: is $\mathcal{T}$ stable with respect to the curvature

$$|\mathcal{T} f_n|_{W^{2,1}(\alpha)} \leq C|f_n|_{W^{2,1}(\mathcal{B}_n(\alpha))} \quad ?$$

- Answer to Q_3 is no...
- ...but up to introducing a discrete curvature $|\cdot|_\star$ for the piecewise affine fonctions, and provided that the numerical E field is bounded in $L^\infty_t(W^{2,\infty}_x)$, $\mathcal{T}$ is stable with respect to

$$\mathcal{E}(f_n, \alpha) := |f_n|_\star(\alpha) + \Delta t \,\mathrm{Vol}(\alpha)|f_n|_{W^{1,\infty}}.$$

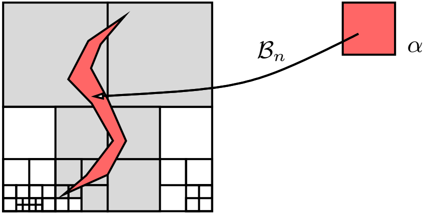
$\rightsquigarrow$ for simplicity, assume that the answer to Q_3 is yes.

Adaptive mesh prediction, II

- Q_2: how big can $|\mathcal{T} f_n|_{W^{2,1}(\alpha)} = |f_n \circ \mathcal{B}_n|_{W^{2,1}(\alpha)}$ be ?
- Answer:

$$|\mathcal{T} f_n|_{W^{2,1}(\alpha)} \leq C |f_n|_{W^{2,1}(\mathcal{B}_n(\alpha))} \leq C \sum_{\beta \in \mathcal{I}(\alpha)} |f_n|_{W^{2,1}(\beta)},$$

where $\mathcal{I}(\alpha)$ contains the cells of M^n that intersect $\mathcal{B}_n(\alpha)$



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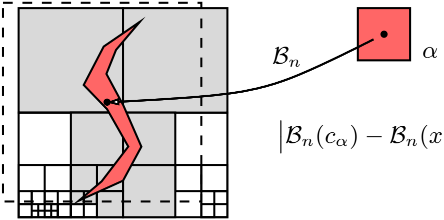


$$\left|\mathcal{B}_n(c_\alpha) - \mathcal{B}_n(x,v)\right| \le C|c_\alpha - (x,v)| \le C 2^{-\ell(\alpha)}$$

Adaptive mesh prediction, III

- Q_1: which cells should be refined in M^{n+1} ?
- Answer: refine α when $\ell(\beta) > \ell(\alpha)$.
- If $\Delta t \leq C(f_0, T)$, the resulting $\tilde{M}^{n+1} := \mathbb{T}[\mathcal{B}_n]M^n$ satisfies:

$$\sup_{\alpha \in \tilde{M}^{n+1}} \#\big(\mathcal{I}(\alpha)\big) \leq C$$

therefore

$$|\mathcal{T} f_n|_{W^{2,1}(\alpha)} \leq C \sum_{\beta \in \mathcal{I}(\alpha)} |f_n|_{W^{2,1}(\beta)} \leq C \sup_{\beta \in M^n} |f_n|_{W^{2,1}(\beta)}.$$

Theorem (CP, Mehrenberger 2005)

M^n is ε-adapted to f_n $\implies$ $\mathbb{T}[\mathcal{B}_n]M^n$ is $C\varepsilon$-adapted to $\mathcal{T} f_n$

the prediction-correction scheme

C P, Mehrenberger *Numer. Math. 2007*

- given (M^n, f_n):
  - ◇ predict a first mesh $\tilde{M}^{n+1} := \mathbb{T}[\mathcal{B}_n]M^n$
  - ◇ perform semi-Lagrangian scheme $\tilde{f}_{n+1} := P_{\tilde{M}^{n+1}}\mathcal{T} f_n$
  - ◇ then correct the mesh $M^{n+1} := \mathbb{A}_\varepsilon(\tilde{f}_{n+1})$
  - ◇ and project again $f_{n+1} := P_{M^{n+1}}\tilde{f}_{n+1}$

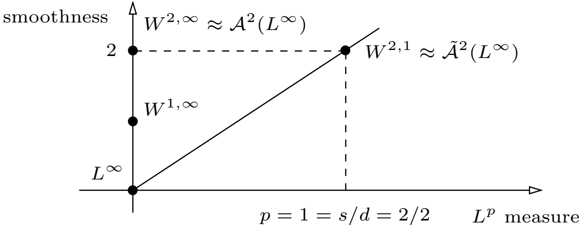
Theorem (CP, Mehrenberger 2005)

$$\|f(t_n) - f_n\|_{L^\infty} \lesssim \Delta t^2 + \varepsilon/\Delta t \sim \varepsilon^{2/3}$$

In addition,

$$\#(\tilde{M}^{n+1}) \lesssim \#(M^n)$$

Convergence rates



- uniform SL scheme: $N := \#(M_h) \sim h^{-2}$

$$f(t) \in W^{2,\infty} \implies \|f(t_n) - f_n\|_{L^\infty} \lesssim \Delta t^2 + h^2/\Delta t \sim h^{4/3} \sim N^{-2/3}$$

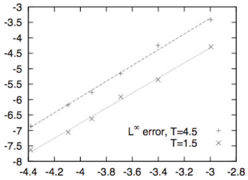
- multi-level adaptive SL scheme

$$f(t) \in W^{1,\infty} \cap W^{2,1} \implies \|f(t_n) - f_n\|_{L^\infty} \lesssim \Delta t^2 + \varepsilon/\Delta t \sim \varepsilon^{2/3}$$

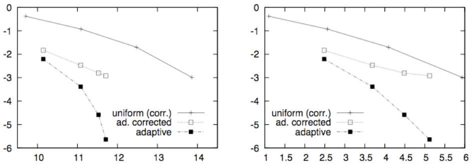
- Estimating $\tilde{N} := \#(M^n)$: still open, but conjecture

$$\tilde{N} \lesssim \varepsilon^{-1} \quad \text{therefore} \quad \|f(t_n) - f_n\|_{L^\infty} \lesssim \tilde{N}^{-2/3}$$

Error vs. time step (top) and complexity (bottom)

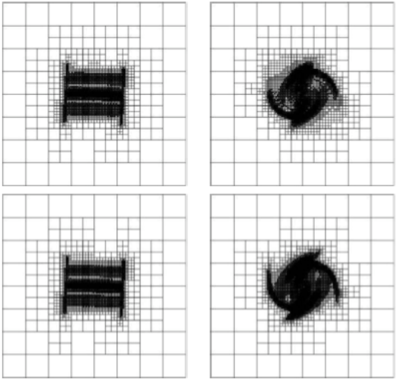


- L^∞ error vs. $\Delta t \sim \varepsilon^{1/3}$ in log-log scale (slopes are around 2.5)



- L^∞ error vs. N (left) and cpu time (right) in log-log scale

Optimality of the adaptive meshes



Work in progress

- parallel versions in higher orders (and up to 4D) have been implemented by M. Mehrenberger, M. Haefele, E. Violard and O. Hoenen
- compare with PIC codes coupled to high order Maxwell solvers
- design anisotropic schemes (using locally refined sparse grids)