Sparse finite element methods

Endre Süli

University of Oxford

Based on joint work with
Christoph Schwab & Radu-Alexandu Todor
ETH Zürich

WPI Vienna, January 2008

Some pointers to the literature

Survey article:

H.-J. Bungartz and M. Griebel. Sparse grids. Acta Numerica, 13:147–123, 2004.

Journal & conference papers:

C. Zenger. Sparse grids. In Parallel algorithms for partial differential equations: Proceedings of the Sixth GAMM-Seminar, Kiel, January 1990.

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Journal & conference papers [continued]:

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T. Von Petersdorff and C. Schwab. Numerical solution of parabolic equations in high dimensions. M2AN Math. Model. Numer. Anal., 38(1):93–127, 2004.

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H. Yserentant. Sparse grid spaces for the numerical solution of the electronic Schrödinger equation. Numerische Mathematik, 101(2):381–389, 2005.

R.A. Todor and C. Schwab. Convergence rates for sparse chaos approximations of elliptic problems with stochastic coefficients. SAM Report. 2006-05. February 2006.

G. Widmer, R. Hiptmair and C. Schwab. Sparse Adaptive Finite Elements for Radiative Transfer. SAM Technical Report. ETH Zürich, January 2007.

C. Schwab, E. Süli, R.-A. Todor. Sparse finite element approximation of high-dimensional transport-dominated diffusion problems. M^2AN (Submitted, 2007).

Scientific motivation

High-dimensional partial differential equations arise in:

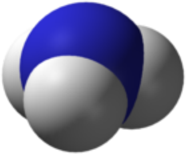
- Stochastic analysis
- Mathematical finance
- Statistical physics
  - Kinetic theory of gases and plasma (Boltzmann and Vlasov equations)
  - Kinetic theory of dilute polymers (degenerate Fokker–Planck equations)
  - Radiative heat transfer equations
- Quantum chemistry: Schrödinger equation

Example 1: Schrödinger equation

High-dimensional PDEs give rise to a major computational challenge.

"One hundred grid points represent a fair resolution for two-point boundary value problems in one space dimension. To obtain the same resolution in three space dimensions, already a million grid points are needed.

The number increases to the unthinkable 10^{60} grid points for equations in 30 dimensions, as in the electronic Schrödinger equation for small molecules



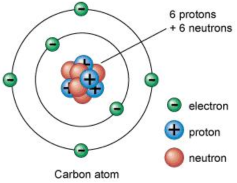
such as water or ammonia."

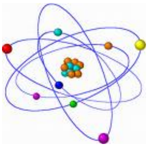
H. Yserentant: Sparse grid spaces for the numerical solution of the electronic Schrödinger equation. Numer. Math. (2005).

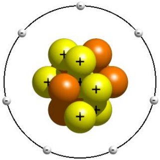
Physically admissible eigenfunctions of the electronic Schrödinger operator

$$H = -\frac{1}{2}\sum_{i=1}^{N}\Delta_i - \sum_{i=1}^{N}\sum_{i=1}^{K}\frac{Z_\nu}{|x_i - a_\nu|} + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{|x_i - x_j|},$$

where $x_1,\ldots,x_N \in \mathbb{R}^3$ are the co-ordinates of N given electrons, a_ν are the co-ordinates of K nuclei, and Z_ν are the charges, are antisymmetric under the exchange of electron coordinates x_i and x_j with indices i and j. (Pauli).

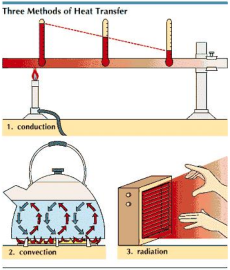






Example 2: Radiative heat transfer equation

Consider the monochromatic radiative heat transfer eq. on a bounded Lipschitz domain $D \subset \mathbb{R}^d$, $d = 2, 3$, without scattering.



We identify a direction s with a point on the unit sphere $\mathbb{S}^d$ and seek the intensity $u(x,s)$:

$$
\begin{aligned}
s \cdot \nabla_x u(x,s) + \kappa(x) u(x,s) &= \kappa(x) f(x), && (x,s) \in D \times \mathbb{S}^d, \\
u(x,s) &= g(x,s), && x \in \partial D, \quad s \cdot n(x) < 0,
\end{aligned}
$$

- $n(x)$ is the unit outer normal to the boundary at $x \in \partial D$,
- $\kappa \geq 0$ is the absorption coefficient,
- $f \geq 0$ is the black-body intensity and $g \geq 0$ is the wall emission.

$\Longrightarrow$ PDE in $d + (d-1) = 2d - 1$ dimensions.

Example 3: Kolmogorov–Fokker–Planck equations

Consider the (Itô) stochastic differential equation:

$$\boxed{\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad X_0 = X.}$$

Here:

- $W = (W^1, \ldots, W^k)$ is a Wiener process w.r.t. a filtration $\{\mathcal{F}_t\,, t \geq 0\}$;
- $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ is Lipschitz continuous $\rightsquigarrow$ dispersion/volatility;
- $b : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous $\rightsquigarrow$ drift.

Define:

- $a := \sigma\sigma^\top : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ $\rightsquigarrow$ diffusion matrix.

Backward Kolmogorov (Fokker–Planck) equation

Theorem

Let the random variable X_t have a density function $(x,t) \mapsto \psi(x,t)$ of class $\mathrm{C}^{2,1}(\mathbb{R}^d \times [0,\infty))$, and let $X_0 = X$ be a square-integrable random variable that is $\mathcal{F}_0$-measurable with density function $\psi_0 \in \mathrm{C}^2(\mathbb{R}^d)$. Then,

$$\partial_t \psi \; + \; \sum_{j=1}^d \frac{\partial}{\partial x_j}(b_j \psi) \; = \; \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \, \partial x_j}(a_{ij} \psi)$$

in $\mathbb{R}^d \times (0,\infty)$ and $\psi(x,0) = \psi_0(x)$ for $x \in \mathbb{R}^d$.

$$\boxed{a(x) = \boldsymbol{\sigma}(x)\boldsymbol{\sigma}^\top(x) \geq 0}$$

Computational challenges:

- PDE non-self-adjoint, transport/drift-dominated, perhaps degenerate
- PDE high-dimensional

Example 4: non-Newtonian fluids

Find $u : \Omega \times (0,\infty) \mapsto \mathbb{R}^3$ and $p : \Omega \times (0,\infty) \mapsto \mathbb{R}$ such that

$$
\begin{array}{rcll}
\partial_t u + (u\cdot\nabla_x)u - \nu\Delta_x u + \nabla_x p &=& f + \nabla_x \cdot \tau & \text{in } \Omega \times (0,\infty), \\
\nabla_x \cdot u &=& 0 & \text{in } \Omega \times (0,\infty), \\
u &=& 0 & \text{on } \partial\Omega \times (0,\infty), \\
u(x,0) &=& u_0(x) & x \in \Omega;
\end{array}
$$

where $\tau(x,t)$ is the symmetric *extra stress tensor*.

Example

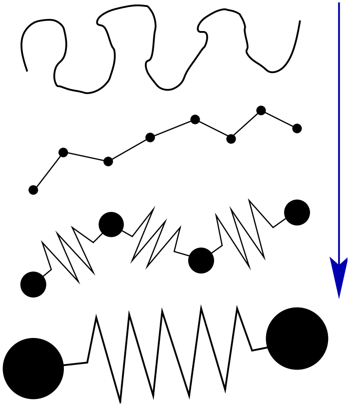
- Algebraic models: $\tau = F(\nabla u)$ — Quasi-Newtonian
- Differential models: $\partial_t \tau + u\cdot\nabla\tau = F(\tau,\nabla u)$ — Oldroyd-B

Non-Newtonian fluids

Gareth McKinley's Non-Newtonian Fluid Dynamics Group, MIT

Jonathan Rothstein's Non-Newtonian Fluids Dynamics Lab, University of Massachusetts

Kinetic polymer models: Kramers chain → dumbbell



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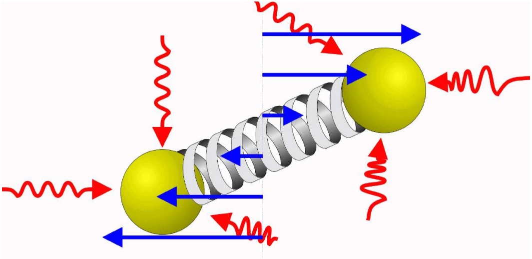
Dumbbell model



$$\begin{cases} \mathrm{d}X_t = u(X_t,t)\,\mathrm{d}t \\ \mathrm{d}Q_t = \big(\nabla_X u(X_t,t)\,Q_t - \frac{1}{2\lambda}F(Q_t)\big)\,\mathrm{d}t + \frac{1}{\sqrt{\lambda}}\,\mathrm{d}W_t\ ; \end{cases}$$

W	vector of independent scalar Wiener processes;
$\lambda = \xi/(4H)$	characterises the elastic property of the fluid;
ξ	drag coefficient and H the spring stiffness;
$F(Q) := U'(\frac{1}{2}\lvert q\rvert^2)\,q$	elastic force acting on the chain due to elongation.

$(x,q,t) \in \mathbb{R}^6 \times \mathbb{R}_{\geq 0} \mapsto \psi(x,q,t) \in \mathbb{R}_{\geq 0}$ is a *probability density function:*

$$\begin{aligned}
\partial_t \psi + (u \cdot \nabla_x)\psi + \nabla_q \cdot \left((\nabla_x u) q \psi - \frac{1}{2\lambda} U' q \psi \right) &= \frac{1}{2\lambda} \Delta_q \psi \quad \text{in } \Omega \times D \times (0,\infty), \\
\psi &= 0 \qquad \text{on } (\Omega \times \partial D) \times (0,\infty), \\
\psi(x,q,0) &= \psi_0(x,q) \quad \text{for } (x,q) \in \Omega \times D.
\end{aligned}$$

Kramers expression for extra stress tensor:

$$\tau(x,t) = k\mu \int_D \psi(x,q,t) \left[U'(\tfrac{1}{2}|q|^2)\, q\, q^\top - \rho(x,t) I \right] \mathrm{d}q, \qquad k, \mu > 0.$$

Example: FENE (finitely extendible nonlinear elastic) potential:

$$U(q) = -\frac{b}{2} \ln\left(1 - \frac{|q|^2}{b}\right), \quad U'(q) = \frac{1}{1 - \frac{|q|^2}{b}}, \qquad q \in D = \{q : |q| < \sqrt{b}\}.$$

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Outline

- PDEs with non-negative characteristic form
- Weak and stabilized variational formulations
- Univariate hierarchical spaces
- Multidimensional sparse tensor product spaces
- Approximability from sparse tensor product spaces
- Stability and convergence of the sparse stabilized FEM

Based on:

C. Schwab, E. Süli, R.-A. Todor: Sparse finite element approximation of high-dimensional transport-dominated diffusion problems. M^2AN (Submitted, 2007).

1. PDEs with non-negative characteristic form

$$\mathcal{L}u := -a : \nabla\nabla u + b \cdot \nabla u + c\,u = f(x), \quad x \in \Omega, \qquad + \text{ BCs},$$

$$\Omega = (0,1)^d \qquad \text{and} \qquad d \gg 1.$$

Assume that $c > 0$, $b \in \mathbb{R}^d$, $a \in \mathbb{R}^{d\times d}$, with $a = a^\top \geq 0$.

Special cases:

- When a is positive definite, the PDE is *elliptic*;
- When $a = 0$ and $b \neq 0$, the PDE is first-order *hyperbolic*;
- When

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{with} \quad \alpha \in \mathbb{R}^{(d-1)\times(d-1)}, \quad \alpha = \alpha^\top > 0$$

and $b = (0,\dots,0,1)^\top \in \mathbb{R}^d$, the PDE is *parabolic*.

2. Weak and stabilized variational formulations

Find $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$:

$$\boxed{\begin{array}{c} \mathcal{L}u \equiv -\nabla\cdot(a\nabla u) + \nabla\cdot(bu) + cu = f \quad \text{in} \quad \Omega, \\[2ex] u = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}} \cup \Gamma_{-}, \qquad n\cdot(a\nabla u) = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}}. \end{array}}$$

$$\zeta^\top a(x)\zeta \geq 0 \quad \forall \zeta \in \mathbb{R}^d, \quad \text{a.e. } x \in \overline{\Omega}.$$

$$\begin{aligned} \Gamma_0 \equiv \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} &= \left\{x \in \Gamma : \; n(x)^\top a(x) n(x) > 0\right\} && \text{(Elliptic boundary)} \\ \Gamma_{-} &= \{x \in \Gamma\backslash\Gamma_0 : \; b(x)\cdot n(x) < 0\} && \text{(Hyperbolic inflow)} \\ \Gamma_{+} &= \{x \in \Gamma\backslash\Gamma_0 : \; b(x)\cdot n(x) \geq 0\} && \text{(Hyperbolic outflow)} \end{aligned}$$

Fichera function: $\quad x \mapsto b(x)\cdot n(x) \quad$ defined on Γ.

Suppose that $v \in \mathrm{H}^1(\Omega)$ with $v|_{\Gamma_{\mathrm{D}}} = 0$. Via formal integration by parts:

$$\int_\Omega a\nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Omega bu \cdot \nabla v \,\mathrm{d}x + \int_\Omega cuv \,\mathrm{d}x \\ - \int_\Gamma (a\nabla u \cdot n) v \,\mathrm{d}s + \int_\Gamma (b \cdot n) uv \,\mathrm{d}s = \int_\Omega fv \,\mathrm{d}x.$$

$$\int_\Gamma (b \cdot n) uv \,\mathrm{d}s = \int_{\Gamma_{\mathrm{N}} \cup \Gamma_+} (b \cdot n) uv \,\mathrm{d}s,$$
$$\int_\Gamma (a\nabla u \cdot n) v \,\mathrm{d}s = \int_{\Gamma \setminus \Gamma_0} ((\nabla u)^\top an) v \,\mathrm{d}s = 0,$$

since $n^\top an = 0$ on $\Gamma \setminus \Gamma_0$ and $a = a^\top \geq 0$ implies that $an = 0$ on $\Gamma \setminus \Gamma_0$.

$\mathcal{V} := \{v \in \mathrm{H}^1(\Omega) : v|_{\Gamma_{\mathrm{D}}} = 0\}, \quad \langle w, v\rangle_\gamma = \int_\gamma |b \cdot n| v w \, \mathrm{d}s, \quad \gamma \subset \Gamma,$

$\mathcal{H} :=$ closure of $\mathcal{V}$ in the norm induced by

$$(w,v)_{\mathcal{H}} := (a\nabla w, \nabla v) + (w,v) + \langle w, v\rangle_{\Gamma_{\mathrm{N}} \cup \Gamma_- \cup \Gamma_+}.$$

Weak formulation: Find $u \in \mathcal{H}$ such that

$$B(u,v) \;=\; \ell(v) \qquad \forall v \in \mathcal{V},$$

where

$$\begin{aligned} B(u,v) &= (a\nabla u, \nabla v) - (u, b \cdot \nabla v) + (cu, v) + \langle u, v\rangle_{\Gamma_{\mathrm{N}} \cup \Gamma_+}, \\ \ell(v) &= (f, v). \end{aligned}$$

Existence of weak solutions:

Oleĭnik & Radkevič (1973)

A special case: $\Omega = (0,1)^d$

Lemma

Each of the sets Γ_0, Γ_+, Γ_- is the union of $(d-1)$-dimensional open faces of Ω. Moreover, each pair of opposite $(d-1)$-dimensional faces of Ω is contained either in the elliptic part Γ_0 of Γ or its complement $\Gamma_- \cup \Gamma_+$, the hyperbolic part of Γ.

We shall assume henceforth that $\Gamma_{\mathrm{N}} = \emptyset$ (i.e. that $\Gamma_0 = \Gamma_{\mathrm{D}}$).

Weak formulation: Find $u \in \mathcal{H}$ such that

$$B(u,v) = \ell(v) \qquad \forall v \in \mathcal{V} = \{v \in \mathrm{H}^1(\Omega) : v|_{\Gamma_0} = 0\},$$

where

$$\begin{aligned} B(u,v) &= (a\nabla u, \nabla v) - (u, b\cdot\nabla v) + (cu, v) + \langle u, v\rangle_{\Gamma_+}, \\ \ell(v) &= (f, v). \end{aligned}$$

Remarks

$u|_{\Gamma_0} = 0$ is imposed strongly, through the definition of $\mathcal{V} \subset \mathcal{H}$,
$u|_{\Gamma_-} = 0$ is imposed weakly, through the definition of $B(\cdot,\cdot)$.

Hence,

$$\bigotimes_{i=1}^{d} \mathrm{H}^1_{(0)}(0,1) := \mathrm{H}^1_{(0)}(0,1) \otimes \cdots \otimes \mathrm{H}^1_{(0)}(0,1) \subset \mathcal{H},$$

where the ith component in the tensor-product is

$$\mathrm{H}^1_{(0)}(0,1) := \begin{cases} \mathrm{H}^1_0(0,1) & \text{if } \mathrm{O}x_i \text{ is an \textit{elliptic} co-ordinate direction,} \\ \\ \mathrm{H}^1(0,1) & \text{if } \mathrm{O}x_i \text{ is a \textit{hyperbolic} co-ordinate direction.} \end{cases}$$

We wish to construct a Galerkin finite element approximation to the boundary-value problem using finite-dimensional subspaces of $\mathcal{H}$ that have *analogous* tensor-product structure.

Stabilization

$$\begin{aligned} -\nabla\cdot(a\nabla u)+\nabla\cdot(bu)+cu=f \quad &\text{in } \Omega,\\ u=0 \quad &\text{on } \Gamma_{\mathrm{D}}\cup\Gamma_{-}. \end{aligned}$$

Perturbed weak formulation: Find $u\in\mathcal{H}$ such that

$$B_\delta(u,v) \;=\; \ell_\delta(v) \qquad \forall v\in\mathcal{V},$$

where

$$\begin{aligned} B_\delta(u,v) &= B(u,v)+\sum_{\alpha\in\mathcal{T}}\delta_\alpha(-\nabla\cdot(a\nabla u)+\nabla\cdot(bu)+cu,\, b\cdot\nabla v)_\alpha\\ \ell_\delta(v) &= \ell(v)+\sum_{\alpha\in\mathcal{T}}\delta_\alpha(f,b\cdot\nabla v)_\alpha, \end{aligned}$$

$\delta_\alpha\geq 0$ — stabilization parameter.

Stabilized finite element method

Find $u_{\mathrm{SD}} \in V_{hp} \subset \mathcal{V}$ such that

$$B_\delta(u_{\mathrm{SD}}, v) \;=\; \ell_\delta(v) \qquad \forall v \in V_{hp}.$$

Streamline-diffusion norm:

$$|||u|||_{\mathrm{SD}} = \left(\|\sqrt{a}\nabla u\|^2_{\mathrm{L}^2(\Omega)} + \|u\|^2_{\mathrm{L}^2(\Omega)} + \|u\|^2_{\Gamma_+ \cup \Gamma_-} + \sum_{\alpha \in \mathcal{T}} \delta_\alpha \|b \cdot \nabla u\|^2_{\mathrm{L}^2(\alpha)} \right)^{\frac{1}{2}}$$

Coercivity and stability: There exist $\delta_0 > 0$, $c_0 > 0$ s.t., for all $\delta \in [0, \delta_0]$,

$$B_\delta(v, v) \geq c_0 |||v|||^2_{\mathrm{SD}} \quad \forall v \in V_{hp}, \qquad |||u_{\mathrm{SD}}|||_{\mathrm{SD}} \leq \frac{1}{c_0} \left(\sum_{\alpha \in \mathcal{T}} (1 + \delta_\alpha) \|f\|^2_{\mathrm{L}^2(\alpha)} \right)^{\frac{1}{2}}.$$

Key observation: for all $v \in V_{hp}$

$$B_\delta(v,v) \geq \int_\Omega |\sqrt{a}\nabla v|^2\,\mathrm{d}x + \int_\Omega \left(c + \tfrac{1}{2}\nabla\cdot b\right)|v|^2\,\mathrm{d}x + \tfrac{1}{2}\int_{\Gamma_{\mathrm{N}}\cup\Gamma_+\cup\Gamma_-} |b\cdot n|\,|u|^2\,\mathrm{d}s$$

$$+\tfrac{1}{2}\sum_{\alpha\in\mathcal{T}}\delta_\alpha\int_\alpha |b\cdot\nabla v|^2\,\mathrm{d}x - \sum_{\alpha\in\mathcal{T}}\delta_\alpha\int_\alpha \left[(c-\nabla\cdot b)^2|v|^2 + |\nabla\cdot(a\nabla v)|^2\right]\mathrm{d}x.$$

Norm-equivalence in finite-dimensional normed linear spaces $\Rightarrow$

Coercivity: There exists $c_0 = c_0(c_*)$ such that

$$B_\delta(v,v) \geq c_0|||v|||^2_{\mathrm{SD}} \qquad \forall v \in V_{hp}.$$

Bibliography [Elliptic theory!]:

- Hughes & Brooks (1979), Johnson & Nävert (1983)
- Baiocchi, Brezzi, Franca (1993), Brezzi & Russo (1994) → RFB/multiscale FEMs
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A brief interlude

Let B_p^d denote the d-dimensional ball of radius L in $\mathbb{R}^d$ in the ℓ_p norm.
Clearly, $B_1^d \subset B_2^d \subset B_\infty^d$.
Question: What are the volumes of the three balls?

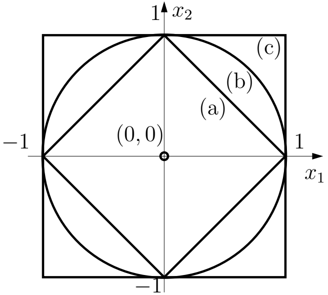


Figure 1: 'Unit circles' in the linear space $\mathbb{R}^2$ with respect to three vector norms: (a) the 1-norm; (b) the 2-norm; (c) the ∞-norm.

Answer:

$$
\begin{aligned}
\mathrm{Vol}(B_\infty^d) &= (2L)^d, \\
\mathrm{Vol}(B_2^d) &= \frac{\pi^{d/2} L^d}{\Gamma(d/2+1)}, \\
\mathrm{Vol}(B_1^d) &= 2^d \frac{\prod_{k=1}^d L}{d!}.
\end{aligned}
$$

Note, in particular, that

$$\frac{\mathrm{Vol}(B_1^d)}{\mathrm{Vol}(B_\infty^d)} = \frac{1}{d!}.$$

If $d = 30$, then $1/d! = 3.77 \cdot 10^{-33}$.
If $d = 100$, then $1/d! = 1.07 \cdot 10^{-158}$.

3. Univariate hierarchical spaces

$\mathcal{T}^{\ell}$:= uniform mesh of spacing $h_\ell = 2^{-\ell}$, $\ell \geq 0$, on $[0,1]$,

$\mathcal{V}^{\ell,p}$:= {all continuous p.w. polynomials of degree p defined on $\mathcal{T}^{\ell}$},

$\mathcal{V}_0^{\ell,p} := \mathcal{V}^{\ell,p} \cap \mathrm{H}_0^1(0,1)$.

The families of spaces $\{\mathcal{V}^{\ell,p}\}_{\ell\geq 0}$ and $\{\mathcal{V}_0^{\ell,p}\}_{\ell\geq 0}$ are nested, i.e.,

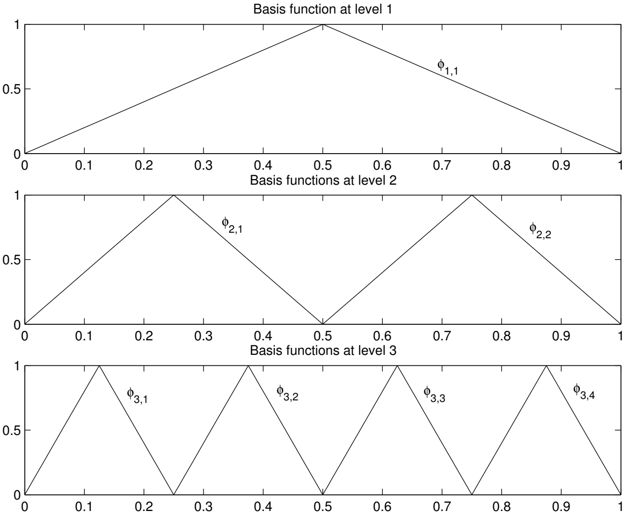
$$\mathcal{V}^{0,p} \subsetneq \mathcal{V}^{1,p} \subsetneq \mathcal{V}^{2,p} \subsetneq \cdots \subsetneq \mathcal{V}^{\ell,p} \subsetneq \cdots \subsetneq \mathrm{H}^1(0,1),$$

and

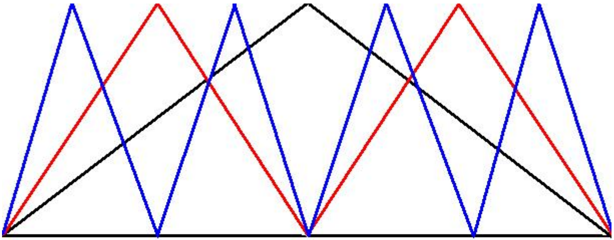
$$\mathcal{V}_0^{0,p} \subsetneq \mathcal{V}_0^{1,p} \subsetneq \mathcal{V}_0^{2,p} \subsetneq \cdots \subsetneq \mathcal{V}_0^{\ell,p} \subsetneq \cdots \subsetneq \mathrm{H}_0^1(0,1).$$

Notation: $\mathcal{V}_{(0)}^{\ell,p}$ is $\mathcal{V}^{\ell,p}$ or $\mathcal{V}_0^{\ell,p}$, as the case may be.

Linear hierarchical basis: $p = 1$



Linear hierarchical basis: $p = 1$



Dimension of the space $= 1 + 2 + \cdots + 2^{L-1} = 2^L - 1$.

A basis-free definition of the subspaces

Consider

$$(P^{\ell,p}u)(x) := u(0) + \int_0^x (\Pi^{\ell,p-1}u')(\xi)\,\mathrm{d}\xi, \qquad P_0^{\ell,p} := P^{\ell,p}|_{\mathrm{H}_0^1(0,1)}.$$

Define

$$\mathcal{V}_{(0)}^{\ell,p} := P_{(0)}^{\ell,p}\mathrm{H}^1_{(0)}(0,1), \qquad \ell \geq 0, \quad p \geq 1.$$

1-d approximation property: Let $u \in \mathrm{H}^{k+1}(0,1) \cap \mathrm{H}^1_{(0)}(0,1)$, $k \geq 1$; then,

$$\|\partial^s(u - P_{(0)}^{\ell,p})u\|_{\mathrm{L}^2(0,1)} \leq \left(\frac{h_\ell}{2}\right)^{t+1-s} \frac{1}{p^{1-s}} \sqrt{\frac{(p-t)!}{(p+t)!}}\, \|\partial^{t+1}u\|_{\mathrm{L}^2(0,1)},$$

where $1 \leq t \leq \min(p,k)$, $h_\ell = 2^{-\ell}$, $\ell \geq 0$, $p \geq 1$, $s \in \{0,1\}$.

Hierarchical decomposition

Incremental projectors:

$$Q_{(0)}^{\ell,p} := \begin{cases} P_{(0)}^{\ell,p} - P_{(0)}^{\ell-1,p}, & \ell \geq 1, \\ P_{(0)}^{0,p}, & \ell = 0. \end{cases}$$

Increment spaces:

$$\mathcal{W}_{(0)}^{\ell,p} := Q_{(0)}^{\ell,p} \mathrm{H}_{(0)}^{1}(0,1), \qquad \ell \geq 0.$$

Now,

$$P_{(0)}^{L,p} = \sum_{\ell=0}^{L} Q_{(0)}^{\ell,p} \qquad \Longrightarrow \qquad \mathcal{V}_{(0)}^{L,p} = \sum_{\ell=0}^{L} \mathcal{W}_{(0)}^{\ell,p}.$$

Proposition

Let X be a vector space; then, there exist nontrivial subspaces X_ℓ, $\ell = 0, \dots, L$, of X such that $X = \bigoplus_{\ell=0}^{L} X_\ell$ if, and only if, there are nonzero linear mappings $q_0, \dots, q_L : X \to X$ such that

(1) $\sum_{\ell=0}^{L} q_\ell = \mathrm{Id}_X$;

(2) $q_{\ell_1} \circ q_{\ell_2} = 0_X$ *for all* $\ell_1, \ell_2 \in \{0, \dots, L\}$, $\ell_1 \neq \ell_2$.

Moreover, each q_ℓ is necessarily a projector and X_ℓ can be chosen to be $\mathrm{Im}(q_\ell)$, $\ell = 0, \dots, L$.

Therefore,

$$Q_{(0)}^{\ell_1,p} Q_{(0)}^{\ell_2,p} = 0, \quad \ell_1 \neq \ell_2, \qquad \Longrightarrow \qquad \mathcal{V}_{(0)}^{L,p} = \bigoplus_{\ell=0}^{L} \mathcal{W}_{(0)}^{\ell,p}.$$

4. Multidimensional sparse tensor-product spaces

Define

$$V_{(0)}^{L,p} := \mathcal{V}_{(0)}^{L,p} \otimes \cdots \otimes \mathcal{V}_{(0)}^{L,p}.$$

Clearly,

$$V_{(0)}^{L,p} = \sum_{|\ell|_\infty \le L} \mathcal{W}_{(0)}^{\ell_1,p} \otimes \cdots \otimes \mathcal{W}_{(0)}^{\ell_d,p}.$$

Sparse tensor-product space

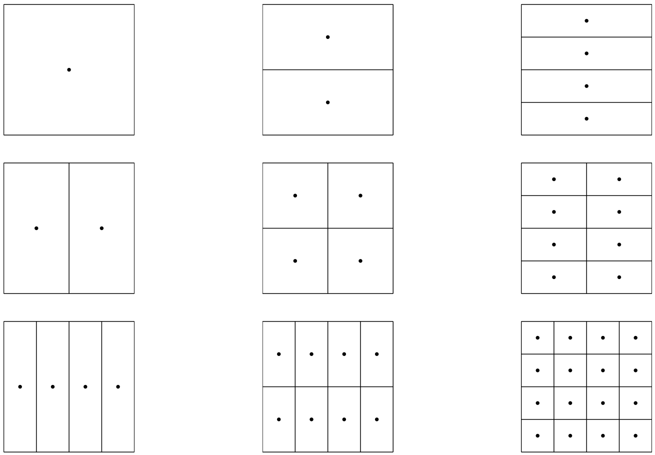
$\longrightarrow$ Babenko (1960), Smolyak (1963), Zenger (1990), Bungartz & Griebel (2004)

$$\hat{V}_{(0)}^{L,p} := \sum_{|\ell|_1 \le L} \mathcal{W}_{(0)}^{\ell_1,p} \otimes \cdots \otimes \mathcal{W}_{(0)}^{\ell_d,p}.$$

Number of DOFs (for p fixed):

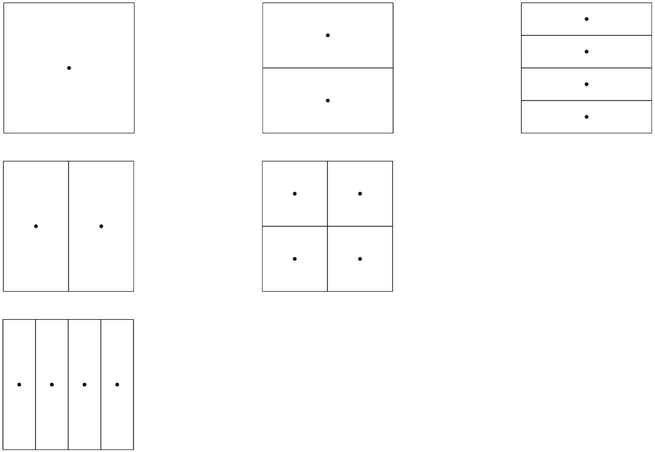
$$\dim V_{(0)}^{L,p} \asymp h_L^{-d}, \qquad \dim \hat{V}_{(0)}^{L,p} \asymp h_L^{-1} |\log_2 h_L|^{d-1}.$$

Supports of basis functions in $V^{L,1}$



Dimension of the space $= (1+2+\cdots+2^{L-1})^d = (2^L-1)^d$.

Supports of basis functions in $\hat{V}^{L,1}$



Dimension of the space $= \sum_{m=1}^{L} \binom{m+d-2}{d-1} 2^{m-1} \sim \dfrac{2^L L^{d-1}}{(d-1)!}$.

Proof:

$$\mathcal{S}(m,k,d) := \{\ell \in \mathbb{N}^d : |\ell|_1 = m, |\ell|_\infty = k\}, \qquad m,k \in \mathbb{N}.$$

Lemma

$$\begin{aligned} \mathcal{S}(m,k,d) &= \emptyset \qquad \forall k > m, \\ \sum_{k=0}^{\infty} |\mathcal{S}(m,k,d)| &= \binom{m+d-1}{d-1}. \end{aligned}$$

$$\begin{aligned} \text{Dimension of the space} &= \sum_{\ell \in \mathbb{N}^d, |\ell|_1 \le L-1} 2^{|\ell|_1} = \sum_{m=0}^{L-1} \sum_{\ell \in \mathbb{N}^d, |\ell|_1 = m} 2^m \\ &= \sum_{m=0}^{L-1} 2^m \sum_{k=0}^{\infty} \sum_{\ell \in \mathbb{N}^d, |\ell|_1 = m, |\ell|_\infty = k} 1 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{L-1} 2^m \sum_{k=0}^{\infty} |\mathcal{S}(m,k,d)| = \sum_{m=0}^{L-1} 2^m \binom{m+d-1}{d-1} \\
&= \frac{1}{(d-1)!} \left(\sum_{m=0}^{L-1} x^{m+d-1} \right)^{(d-1)} \Bigg|_{x=2} \\
&= \frac{1}{(d-1)!} \left(x^{d-1} \frac{x^L - 1}{x-1} \right)^{(d-1)} \Bigg|_{x=2} \\
&= \frac{1}{(d-1)!} \sum_{m=0}^{d-1} \binom{d-1}{m} \cdot (x^{L+d-1} - x^{d-1})^{(m)} \cdot \left(\frac{1}{x-1} \right)^{(d-1-m)} \Bigg|_{x=2} \\
&= 2^L \sum_{m=0}^{d-1} \binom{L+d-1}{m} (-2)^{d-1-m} + (-1)^d \\
&\sim \frac{2^L L^{d-1}}{(d-1)!} = \frac{1}{(d-1)!} h_L^{-1} |\log_2 h_L|^{d-1}.
\end{aligned}
$$

5. Approximability from sparse tensor product spaces

Full tensor-product projector:

$$P_{(0)}^{L,p} = \sum_{|\ell|_\infty \le L} Q_{(0)}^{\ell_1,p} \otimes \cdots \otimes Q_{(0)}^{\ell_d,p} : \bigotimes_{i=1}^{d} \mathrm{H}^1_{(0)}(0,1) \to V_{(0)}^{L,p}, \qquad \ell = (\ell_1, \ldots, \ell_d).$$

Sparse tensor-product projector:

$$\hat{P}_{(0)}^{L,p} = \sum_{|\ell|_1 \le L} Q_{(0)}^{\ell_1,p} \otimes \cdots \otimes Q_{(0)}^{\ell_d,p} : \bigotimes_{i=1}^{d} \mathrm{H}^1_{(0)}(0,1) \to \hat{V}_{(0)}^{L,p}, \qquad \ell = (\ell_1, \ldots, \ell_d).$$

Define

$$|u|_{\mathcal{H}^{t+1}(\Omega)} := \max_{s\in\{0,1\}} \max_{1\le k\le d} \left(\max_{\substack{J\subseteq\{1,2,\ldots,d\} \\ |J|=k}} |u|_{\mathrm{H}^{t+1,s,J}(\Omega)} \right).$$

Theorem

Let $\Omega = (0,1)^d$, $s \in \{0,1\}$, $k \geq 1$, $p \geq 1$ *be given. For* $1 \leq t \leq \min\{p,k\}$, *there exist* $\underline{c}_{p,t} > 0$, $\kappa_{(0)}(p,t,s,L) > 0$, *independent of* d, *such that, for any* $u \in \mathcal{H}^{k+1}(\Omega)$ *and all* $L \geq 1$ *and* $d \geq 2$, *we have*

$$|u - \hat{P}^{L,p}_{(0)} u|_{\mathrm{H}^s(\Omega)} \leq d^{1+\frac{s}{2}} \underline{c}_{p,t} (\kappa_{(0)}(p,t,s,L))^{d-1+s} 2^{-(t+1-s)L} |u|_{\mathcal{H}^{t+1}(\Omega)}.$$

$$\kappa_{(0)}(p,t,s,L) := \begin{cases} \tilde{c}_{p,0,t}(L+1)\mathrm{e}^{1/(L+1)} + \hat{c}_{p,0,(0)}, & s = 0, \\ 2\tilde{c}_{p,0,t} + \hat{c}_{p,0,(0)}, & s = 1. \end{cases}$$

$$\hat{c}_{p,0,(0)} := |Q^{0,p}_{(0)}|_{\mathcal{B}(\mathrm{H}^1_{(0)}(0,1),\mathrm{L}^2(0,1))}.$$

The constants

$$\kappa_{(0)}(p,t,s,L) := \begin{cases} \tilde{c}_{p,0,t}(L+1)\mathrm{e}^{1/(L+1)} + \hat{c}_{p,0,(0)}, & s=0, \\ 2\tilde{c}_{p,0,t} + \hat{c}_{p,0,(0)}, & s=1. \end{cases}$$

For $p \geq 1$, $t \in \mathbb{N}$, $1 \leq t \leq p$, $s \in \{0,1\}$:

$$\tilde{c}_{p,0,t} = \left(1 + \frac{1}{2^{t+1-s}}\right)\frac{1}{p}\sqrt{\frac{(p-t)!}{(p+t)!}}, \qquad \hat{c}_{p,0,(0)} = \frac{1}{\pi}.$$

Refined values for $p=1$:

$$\tilde{c}_{1,0,1} = \frac{1}{3}, \qquad \hat{c}_{1,0,0} = 0.$$

Tracking the constants

Remark (A)

If $\Gamma = \Gamma_0$ *(elliptic problem) and* $s = 1$ $(\mathrm{H}^1(\Omega)$ *seminorm error), then*

$$\kappa_0(p,p,1,L) < 1 \qquad \forall p \geq 1, \quad L \geq 1.$$

Therefore the factor $(\kappa_0(p,p,1,L))^{d-1+s}$ *decays exponentially as* $d \to \infty$.

M. Griebel (CUP, 2006: Proc. Found. Comp. Math. Santander, Spain, 2005), $p = 1$, $s = 1$, *under stronger,* $\mathrm{W}^{2,\infty}(\Omega)$, *regularity on* u, *for* $-\Delta u = f$ *with* $u|_\Gamma = 0$.

Remark (B)

If $s = 0$ *(i.e. for* $\mathrm{L}^2(\Omega)$ *norm error), no* $|\log_2 h_L|^{d-1}$ *term, if*

$$p = 2 \text{ and } L \leq 3, \qquad p = 3 \text{ and } L \leq 49, \qquad p \geq 4 \text{ and } L \leq 528.$$

Remark (C)

If $s = 0$ *and*

$$\gamma_{(0)}(p,t) := \tilde{c}_{p,0,t} 2^{t+1}/(2^{t+1}-1) + \hat{c}_{p,0,(0)} < 1,$$

then there exists a positive constant $c_{t,p}$*, independent of* L *and* d*, such that* $\kappa_{(0)}(p,t,0,L) < 1$ *for all* $L \geq 1$ *and* $d \geq 2$ *satisfying* $L+1 \leq c_{t,p}(d-1)$*.*

If $\Gamma = \Gamma_0$*, then* $\gamma_0(p,p) < 1$ *for all* $p \geq 1$*. Also,*

$$\kappa_0(p,p,0,L) < 1$$

whenever

$$L+1 \leq c_{p,p}(d-1),$$

where

$$c_{1,1} = 0.6, \quad c_{2,2} = 0.71, \quad c_{3,3} = 1.846, \quad c_{4,4} = 2.161, \ldots.$$

Remark (D)

If $\Gamma_0 \subsetneq \Gamma$ (*i.e. hyperbolic boundary* $\Gamma_- \cup \Gamma_+ \neq \emptyset$), *then*

- *for* $s = 1$, *i.e. for* $\mathrm{H}^1(\Omega)$ *seminorm error:*

$$\kappa_{(0)}(p,p,1,L) < 1 \qquad \textit{when} \qquad \begin{cases} p = 2 & \textit{and} \quad d \leq 7, \\ p = 3 & \textit{and} \quad d \leq 71, \\ p = 4 & \textit{and} \quad d \leq 755. \end{cases}$$

- *for* $s = 0$, *i.e. for* $\mathrm{L}^2(\Omega)$ *error, the worst-case scenario is:*

$$\kappa_{(0)}(p,p,0,L) \leq (L+1)^{d-1} \kappa_*^{d-1},$$

where

$$\kappa_* = \frac{1}{L+1} + \frac{2}{p\sqrt{(2p)!}} < 1$$

for $L \geq 1$ *and* $p \geq 2$.

Technical ingredients of the proof

`1. First ingredient: tensorization of seminorms`

Proposition

Let $(\mathrm{H}_i, \langle\cdot,\cdot\rangle_{\mathrm{H}_i})$, $(\mathrm{K}_i, \langle\cdot,\cdot\rangle_{\mathrm{K}_i})$, $(\tilde{\mathrm{H}}_i, \langle\cdot,\cdot\rangle_{\tilde{\mathrm{H}}_i})$, $(\tilde{\mathrm{K}}_i, \langle\cdot,\cdot\rangle_{\tilde{\mathrm{K}}_i})$ *for* $i = 1,2$ *be separable Hilbert spaces.*

Let $T_i \in \mathcal{B}(\mathrm{H}_i, \mathrm{K}_i), \tilde{T}_i \in \mathcal{B}(\tilde{\mathrm{H}}_i, \tilde{\mathrm{K}}_i)$ *and* $Q_i \in \mathcal{B}(\mathrm{H}_i, \tilde{\mathrm{H}}_i)$ *be bounded linear operators, and assume that* $\|\tilde{T}_i Q_i v_i\|_{\tilde{\mathrm{K}}_i} \leq c_i \|T_i v_i\|_{\mathrm{K}_i} \quad \forall v_i \in \mathrm{H}_i$, $i = 1,2$.

Then

$$\|(\tilde{T}_1 \otimes \tilde{T}_2)(Q_1 \otimes Q_2)u\|_{\tilde{\mathrm{K}}_1 \otimes \tilde{\mathrm{K}}_2} \leq c_1 c_2 \|(T_1 \otimes T_2)u\|_{\mathrm{K}_1 \otimes \mathrm{K}_2} \qquad \forall u \in \mathrm{H}_1 \otimes \mathrm{H}_2.$$

In terms of an abbreviated notation:

$$|Q_i|_{(T_i, \tilde{T}_i)} \leq c_i, \quad i = 1,2 \quad \Rightarrow \quad |Q_1 \otimes Q_2|_{(T_1 \otimes T_2, \tilde{T}_1 \otimes \tilde{T}_2)} \leq c_1 c_2.$$

$$T_i \in \mathcal{B}(\mathrm{H}_i, \mathrm{K}_i),\ \tilde{T}_i \in \mathcal{B}(\tilde{\mathrm{H}}_i, \tilde{\mathrm{K}}_i),\ Q_i \in \mathcal{B}(\mathrm{H}_i, \tilde{\mathrm{H}}_i), \qquad \|Q_i\|_{T_i, \tilde{T}_i} \le c_i, \quad i = 1, 2.$$

$$\begin{array}{ccc} \mathrm{H}_1 & \xrightarrow{T_1} & \mathrm{K}_1 \\ \Big\downarrow Q_1 & & \\ \tilde{\mathrm{H}}_1 & \xrightarrow{\tilde{T}_1} & \tilde{\mathrm{K}}_1 \end{array} \qquad \qquad \begin{array}{ccc} \mathrm{H}_2 & \xrightarrow{T_2} & \mathrm{K}_2 \\ \Big\downarrow Q_2 & & \\ \tilde{\mathrm{H}}_2 & \xrightarrow{\tilde{T}_2} & \tilde{\mathrm{K}}_2 \end{array}$$

$$\begin{array}{ccc} \mathrm{H}_1 \otimes \mathrm{H}_2 & \xrightarrow{T_1 \otimes T_2} & \mathrm{K}_1 \otimes \mathrm{K}_2 \\ \Big\downarrow Q_1 \otimes Q_2 & & \\ \tilde{\mathrm{H}}_1 \otimes \tilde{\mathrm{H}}_2 & \xrightarrow{\tilde{T}_1 \otimes \tilde{T}_2} & \tilde{\mathrm{K}}_1 \otimes \tilde{\mathrm{K}}_2 \end{array}$$

$$\|Q_1 \otimes Q_2\|_{T_1 \otimes T_2, \tilde{T}_1 \otimes \tilde{T}_2} \le c_1 c_2.$$

2. Second ingredient: Explicit bounds on lattice sums

Lemma

Suppose that $d, m \in \mathbb{N}_{>0}$ *and* $x > 1$. *Then,*

$$d \cdot x^m \leq \sum_{\ell \in \mathbb{N}^d,\ |\ell|_1 = m} x^{|\ell|_\infty} \leq d \left(1 + \frac{1}{x-1}\right)^{d-1} \cdot x^m.$$

Lemma

For $L, d \in \mathbb{N}_{>0}$, $\alpha, \beta > 0$, *and* $x \geq 2$ *define*

$$A(L,d,x) := \sum_{\substack{\ell \in \mathbb{N}^d \\ |\ell|_1 > L}} x^{-|\ell|_1},$$

$$B(L,d,x,\alpha,\beta) := \sum_{k=1}^{d} \binom{d}{k} \alpha^k \beta^{d-k} A(L,k,x).$$

Then

$$B(L,d,x,\alpha,\beta) \leq \frac{\alpha \mathrm{e} d x}{x-1} \cdot (\alpha(L+1)\mathrm{e}^{1/(L+1)} + \beta)^{d-1} \cdot x^{-(L+1)}.$$

Lemma

For $L, d \in \mathbb{N}_{>0}$*,* $\alpha, \beta > 0$*, and* $x \geq 2$ *define*

$$A(L,d,x) := \sum_{\substack{\ell \in \mathbb{N}^d \\ |\ell|_1 > L}} x^{-|\ell|_1},$$

$$B(L,d,x,\alpha,\beta) := \sum_{k=1}^{d} \binom{d}{k} \alpha^k \beta^{d-k} A(L,k,x).$$

If $\gamma := \alpha \cdot x/(x-1) + \beta < 1$*, then there exists* $c_{1,x,\gamma} > 0, c_{2,x,\gamma} \in (0,1)$ *such that*

whenever $\quad d \geq 2 \quad$ *and* $\quad L + 1 \leq c_{1,x,\gamma}(d-1)$

we have

$$B(L,d,x,\alpha,\beta) \leq \frac{\alpha\, d\, x}{x-1} \cdot c_{2,x,\gamma}^{d-1} \cdot x^{-(L+1)}.$$

Proof of the Theorem: $[s=0]$

For $u \in \mathrm{C}^{\infty}_{(0)}(\bar{\Omega}) \subset \mathrm{L}^2(\Omega)$, the following identity holds in $\mathrm{L}^2(\Omega)$:

$$\begin{aligned}
\|u - \hat{P}^{L,p}_{(0)} u\|_{\mathrm{L}^2(\Omega)} &\leq \sum_{\ell \in \mathbb{N}^d,\ |\ell|_1 > L} \left\| \left(Q^{\ell_1,p}_{(0)} \otimes \cdots \otimes Q^{\ell_d,p}_{(0)} \right) u \right\|_{\mathrm{L}^2(\Omega)} \\
&= \sum_{k=1}^{d} \sum_{\substack{I \subset \{1,2,\dots,d\} \\ |I|=k}} \sum_{\substack{\ell \in \mathbb{N}^d,\ |\ell|_1 > L \\ \mathrm{supp}(\ell) = I}} \left\| \left(Q^{\ell_1,p}_{(0)} \otimes \cdots \otimes Q^{\ell_d,p}_{(0)} \right) u \right\|_{\mathrm{L}^2(\Omega)} .
\end{aligned}$$

Now, for any $\ell \in \mathbb{N}^d$ with $I = \mathrm{supp}(\ell)$ and$|I| = k$:

$$\begin{aligned}
&\left\| \left(Q^{\ell_1,p}_{(0)} \otimes \cdots \otimes Q^{\ell_d,p}_{(0)} \right) u \right\|^2_{\mathrm{L}^2(\Omega)} \\
&\quad \leq \left\{ \prod_{j \in I} |Q^{\ell_j,p}_{(0)}|^2_{(\partial^{t+1}, \mathrm{Id}_{\mathrm{L}^2(0,1)})} \right\} |Q^{0,p}_{(0)}|^{2(d-k)}_{(\mathrm{Id}_{\mathrm{H}^1_{(0)}(0,1)}, \mathrm{Id}_{\mathrm{L}^2(0,1)})} |u|^2_{\mathrm{H}^{t+1,0,I}(\Omega)} \\
&\quad = \tilde{c}^{2k}_{p,0,t} \hat{c}^{2(d-k)}_{p,0,(0)} 2^{-2(t+1)|\ell|_1} |u|^2_{\mathrm{H}^{t+1,0,I}(\Omega)} .
\end{aligned}$$

Summing this bound over all $I \subseteq \{1,2,\ldots,d\}$ with $|I| = k$ implies

$$\begin{aligned} \|u - \hat{P}^{L,p}_{(0)} u\|_{\mathrm{L}^2(\Omega)} &\leq \sum_{k=1}^{d} \binom{d}{k} \tilde{c}^{k}_{p,0,t} \hat{c}^{d-k}_{p,0,(0)} \left\{ \sum_{\substack{\ell \in \mathbb{N}^k \\ |\ell|_1 > L}} 2^{-(t+1)|\ell|_1} \right\} \\ &\quad \times \max_{1 \leq k \leq d} \left(\max_{\substack{I \subset \{1,2,\ldots,d\} \\ |I| = k}} |u|_{\mathrm{H}^{t+1,0,I}(\Omega)} \right). \end{aligned}$$

Using the lattice sum lemmas 2 and 3 with $x := 2^{t+1} \geq 2$ for $t \geq 0$, $\alpha := \tilde{c}_{p,0,t}$, and $\beta := \hat{c}_{p,0,(0)}$ we obtain

$$\|u - \hat{P}^{L,p}_{(0)} u\|_{\mathrm{L}^2(\Omega)} \leq 2d\,\mathrm{e}\,\tilde{c}_{p,0,t} \cdot \kappa_{(0)}(p,t,0,L)^{d-1} \cdot 2^{-(t+1)(L+1)} |u|_{\mathcal{H}^{t+1}(\Omega)}$$

where

$$\kappa_{(0)}(p,t,0,L) := \tilde{c}_{p,0,t}(L+1)\mathrm{e}^{1/(L+1)} + \hat{c}_{p,0,(0)}, \qquad p \geq 1, \quad 1 \leq t \leq p, \quad L \geq 1.$$

Hence the required bound for $s=0$, with $\underline{c}_{p,t} = 2^{-t}\mathrm{e}\,\tilde{c}_{p,0,t}$.

Further, if

$$\gamma_{(0)}(t,p) := \tilde{c}_{p,0,t}2^{t+1}/(2^{t+1}-1) + \hat{c}_{p,0,(0)} < 1,$$

then there exists a constant $c_{t,p} > 0$, independent of L and d, such that $\kappa_{(0)}(p,t,0,L) < 1$ for all $L \geq 1$ and $d \geq 2$ satisfying $L+1 \leq c_{t,p}(d-1)$. $\square$

6. Stability and convergence of the sparse stabilized FEM

Theorem

Suppose that

$$0 \leq \delta_L \leq \min\left(\frac{h_L^2}{12dp^4|\sqrt{a}|^2}, \frac{1}{c}\right).$$

Then,

$$\forall v_h \in \hat{V}_{(0)}^{L,p} : \qquad B_\delta(v_h, v_h) \geq \frac{1}{2} |||v_h|||_{\mathrm{SD}}^2.$$

Now, fix

$$\delta_L := K_\delta \cdot \min\left(\frac{h_L^2}{12dp^4|\sqrt{a}|^2}, \frac{h_L}{|b|}, \frac{1}{c}\right),$$

with $K_\delta \in \mathbb{R}_{>0}$ a constant, independent of h_L and d.

Theorem

Let $f \in \mathrm{L}^2(\Omega)$, $\Omega = (0,1)^d$, $u \in \mathcal{H}^{k+1}(\Omega) \cap \mathrm{H}^2(\Omega) \cap \bigotimes_{i=1}^{d} \mathrm{H}^1_{(0)}(0,1)$, $k \geq 1$, *and let the stabilization parameter* δ_L *be as above.*

If $p \geq 1$, $1 \leq t \leq \min(p,k)$, $h = h_L = 2^{-L}$ *and* $L \geq 1$, *then*

$$|||u - u_h|||_{\mathrm{SD}} \leq C_{p,t}\, d^2 \max\{(2-p)_+, \kappa_{(0)}(p,t,0,L)^{d-1}, \kappa_{(0)}(p,t,1,L)^d\} \times \left(|\sqrt{a}| h_L^t + |b|^{\frac{1}{2}} h_L^{t+\frac{1}{2}} + c^{\frac{1}{2}} h_L^{t+1} \right) |u|_{\mathcal{H}^{t+1}(\Omega)}.$$

Sketch of the proof

Let $h = h_L = 2^{-L}$.

$$|||u - u_h|||_{\mathrm{SD}} \leq |||u - \hat{P}^{L,p}_{(0)} u|||_{\mathrm{SD}} + |||\hat{P}^{L,p}_{(0)} u - u_h|||_{\mathrm{SD}}.$$

The first term on the right is bounded using the approximation Thm from Sec. 5. Further, by coercivity of B_δ on $\hat{V}^{L,p}_{(0)}$ and Galerkin orthogonality,

$$\begin{aligned} \frac{1}{2}|||\hat{P}^{L,p}_{(0)} u - u_h|||^2_{\mathrm{SD}} &\leq B_\delta(\hat{P}^{L,p}_{(0)} u - u_h, \hat{P}^{L,p}_{(0)} u - u_h) \\ &= -B_\delta(u - \hat{P}^{L,p}_{(0)} u, \hat{P}^{L,p}_{(0)} u - u_h) \end{aligned}$$

since

$$B_\delta(u - u_h, \hat{P}^{L,p}_{(0)} u - u_h) = 0.$$

Roughly (and not entirely correctly; the precise argument is *much* more involved):

$$\left| B_\delta(u - \hat{P}^{L,p}_{(0)} u, \hat{P}^{L,p}_{(0)} u - u_h) \right| \leq Const. |||u - \hat{P}^{L,p}_{(0)} u|||_{\mathrm{SD}}\, |||\hat{P}^{L,p}_{(0)} u - u_h|||. \qquad \square$$

How about $|u|_{\mathcal{H}^{t+1}(\Omega)}$?

Consider, on $\Omega = (0,1)^d$, the PDE

$$-a : \nabla\nabla u + b \cdot \nabla u + cu = f(x), \qquad x \in \Omega,$$

with $f \in \mathrm{L}^2(\Omega)$, constant coefficients $a \in \mathbb{R}^{d \times d}$, $b \in \mathbb{R}^d$, and $c \in \mathbb{R}_{>0}$, $a^\top = a \geq 0$, subject to *periodic* boundary conditions.

Recall that

$$|u|_{\mathcal{H}^{t+1}(\Omega)} := \max_{s \in \{0,1\}} \max_{1 \leq k \leq d} \left(\max_{\substack{J \subseteq \{1,2,\ldots,d\} \\ |J|=k}} |u|_{\mathrm{H}^{t+1,s,J}(\Omega)} \right).$$

We shall therefore begin by considering, for $s \in \{0,1\}$, $k \in \{1,\ldots,d\}$ and $J \subset \{1,\ldots,d\}$, with $|J| = k$,

$$|u|_{\mathrm{H}^{t+1,s,J}(\Omega)}.$$

$$u = \sum_{m \in \mathbb{Z}^d} \hat{u}_m \mathrm{e}^{2\pi \mathrm{i} m \cdot x}, \qquad f = \sum_{m \in \mathbb{Z}^d} \hat{f}_m \mathrm{e}^{2\pi \mathrm{i} m \cdot x}.$$

Substituting these into the PDE yields

$$[m^\top a m + i(b \cdot m) + c]\hat{u}_m = \hat{f}_m \qquad \forall m \in \mathbb{Z}^d.$$

Hence,

$$|\hat{u}_m|^2 = \frac{|\hat{f}_m|^2}{(m^\top a m + c)^2 + |b \cdot m|^2} \qquad \forall m \in \mathbb{Z}^d.$$

Since $a \geq 0$ and $c > 0$, it follows that

$$|\hat{u}_m|^2 \leq \frac{1}{c^2} |\hat{f}_m|^2 \qquad \forall m \in \mathbb{Z}^d.$$

Assume without loss of generality that $J = \{1,\ldots,k\}$, where $1 \le k \le d$. Hence,

$$|u|^2_{\mathrm{H}^{t+1,s,J}(\Omega)} = \sum_{m\in\mathbb{Z}^d} (2m_1\pi)^{2(t+1)} \cdots (2m_k\pi)^{2(t+1)} (2m_{k+1})^{2s} \cdots (2m_d)^{2s} |\hat{u}_m|^2.$$

Therefore,

$$|u|^2_{\mathrm{H}^{t+1,s,J}(\Omega)} \le \frac{1}{c^2} \sum_{m\in\mathbb{Z}^d} (2m_1\pi)^{2(t+1)} \cdots (2m_k\pi)^{2(t+1)} (2m_{k+1})^{2s} \cdots (2m_d)^{2s} |\hat{f}_m|^2.$$

Equivalently,

$$|u|^2_{\mathrm{H}^{t+1,s,J}(\Omega)} \le c^{-2} |f|^2_{\mathrm{H}^{t+1,s,J}(\Omega)}.$$

Therefore,

$$|u|^2_{\mathcal{H}^{t+1}(\Omega)} \le c^{-2} |f|^2_{\mathcal{H}^{t+1}(\Omega)}.$$

For example, if $f(x_1,\dots,x_d)=f_1(x_1)\cdots f_d(x_d)$, then

$$|f|_{\mathrm{H}^{t+1,s,J}(\Omega)} = |f_1|_{\mathrm{H}^{t+1}(0,1)}\cdots|f_k|_{\mathrm{H}^{t+1}(0,1)}|f_{k+1}|_{H^s(0,1)}\cdots|f_d|_{\mathrm{H}^s(0,1)}.$$

Let

$$\alpha_0 = \max_{1\le k\le d}\max_{s\in\{0,1\}}\{|f_k|_{\mathrm{H}^{t+1}(0,1)}, \|f_k\|_{\mathrm{H}^s(0,1)}\}.$$

Then,

$$|f|_{\mathcal{H}^{t+1}(\Omega)} \le \alpha_0^d,$$

and therefore,

$$|u|_{\mathcal{H}^{t+1}(\Omega)} \le c^{-1}\alpha_0^d.$$

Example

$$f(x_1,\dots,f_d) = \frac{1}{(2\pi)^{d(t+1)}}\prod_{k=1}^{d}\sin 2\pi x_k.$$

$$|f|_{\mathcal{H}^{t+1}(\Omega)} \le 1.$$

Conclusions

1. For 2nd-order PDEs with non-negative characteristic form on $\Omega = (0,1)^d$, we developed a stabilized variational formulation.
2. Formulation stable on sparse tensor-product space, of meshwidth $h = h_L$ and polynomial degree $p \geq 1$, independent of:
   - ⋆ mesh Péclet number;
   - ⋆ anisotropy in basis functions;
   - ⋆ degeneracy of elliptic part.
3. error analysis shows that the constant decreases exponentially as $d \to \infty$ (substantially generalizing M. Griebel (2006) from $p = 1$ and Dirichlet b.v.p. for $-\Delta u = f$, to $p > 1$, any sparse basis, and second-order PDEs with non-negative characteristic form).
4. We have identified a number of preasymptotic regimes where there is no $|\log_2 h_L|^{d-1}$ term in the error bound.

Comments

The statements above presuppose that

$$|u|_{\mathcal{H}^{t+1}(\Omega)} := \max_{s\in\{0,1\}} \max_{1\leq k\leq d} \left(\max_{\substack{J\subseteq\{1,2,\ldots,d\} \\ |J|=k}} |u|_{\mathrm{H}^{t+1,s,J}(\Omega)} \right)$$

is bounded as $d \to \infty$.

A poorly understood question:
analysis of regularity and growth of norms of solutions of high-dimensional PDEs in spaces of functions with square-integrable mixed derivatives.

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