Volume Effects in Chemotaxis

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supported by NSERC

University of Alberta

Outline

- (1) The Classical Chemotaxis Equations

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- (2) Volume Effects

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- (2) Volume Effects
- (2.a) Volume Filling

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- (2.b) Quorum Sensing
- (2.c) Finite Sampling Radius

(1) The Classical Chemotaxis Model

$$
\begin{aligned}
u_t &= D_u \Delta u - \chi_0 \nabla \cdot \{u \nabla v\} \\
v_t &= D_v \Delta v + f(u,v)
\end{aligned}
$$

$u(t,x)$: particle density

$v(t,x)$: concentration of chemical signal

D_u: diffusion coefficient, χ_0 *chemotactic sensitivity*

$f(u,v)$: production and consumption of signal.

References

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- *Horstmann 2004:* Review.
- *Hillen + Potapov 2004:* spikes in 1-D.

(2) Volume Effects

Classification of volume effects in Hillen and Hadeler 2004:

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- (a) Volume Filling

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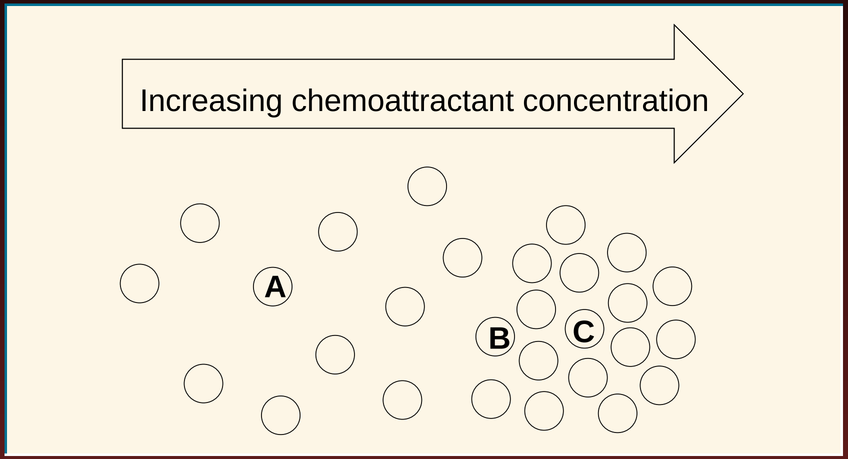
(a) Volume Filling

(w. K. Painter)



(a) Volume Filling

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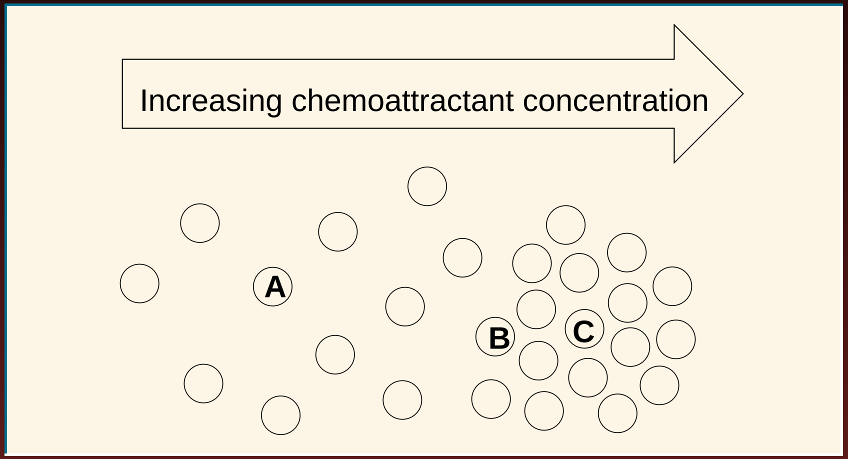

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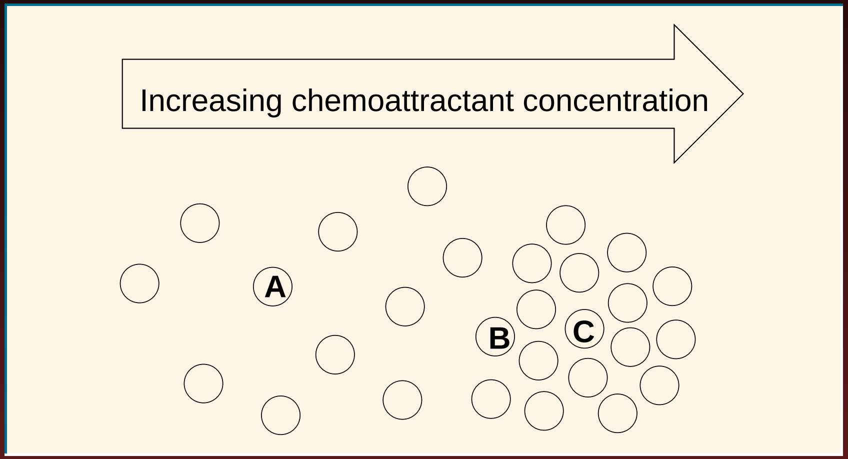

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Assumption

$$q(U_{\max}) = 0 \text{ and } q(u) \geq 0 \text{ for all } 0 \leq u < U_{\max}$$

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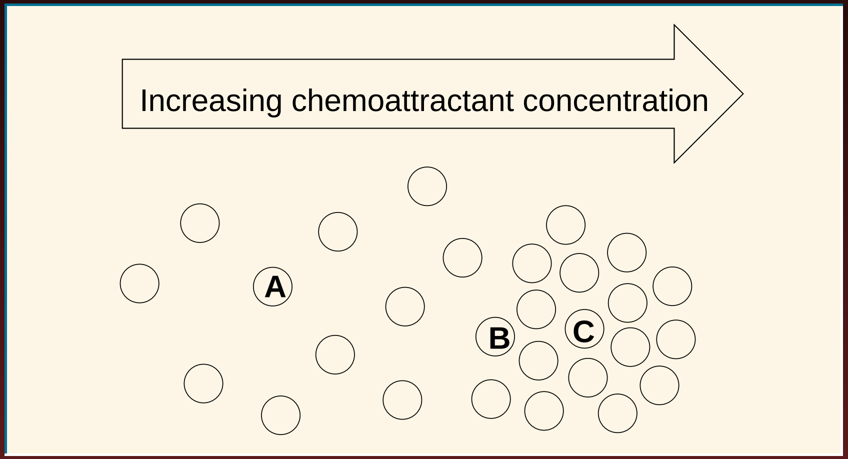

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$$q(U_{\text{max}}) = 0 \text{ and } q(u) \geq 0 \text{ for all } 0 \leq u < U_{\text{max}}$$

Standard example: $U_{\text{max}} = 1, \qquad q(u) = 1 - u.$

The Volume Filling Model

$$
\begin{aligned}
u_t &= \nabla(D_u(q(u) - q'(u)u)\nabla u - q(u)u\chi(v)\nabla v) \\
v_t &= D_v \Delta v + f(u,v)
\end{aligned}
$$

Complete Picture [1]-[7]

- [1] *Hillen + Painter 2000:*
  First mention of the volume filling model; proof of global existence for special cases; numerical pattern formation.

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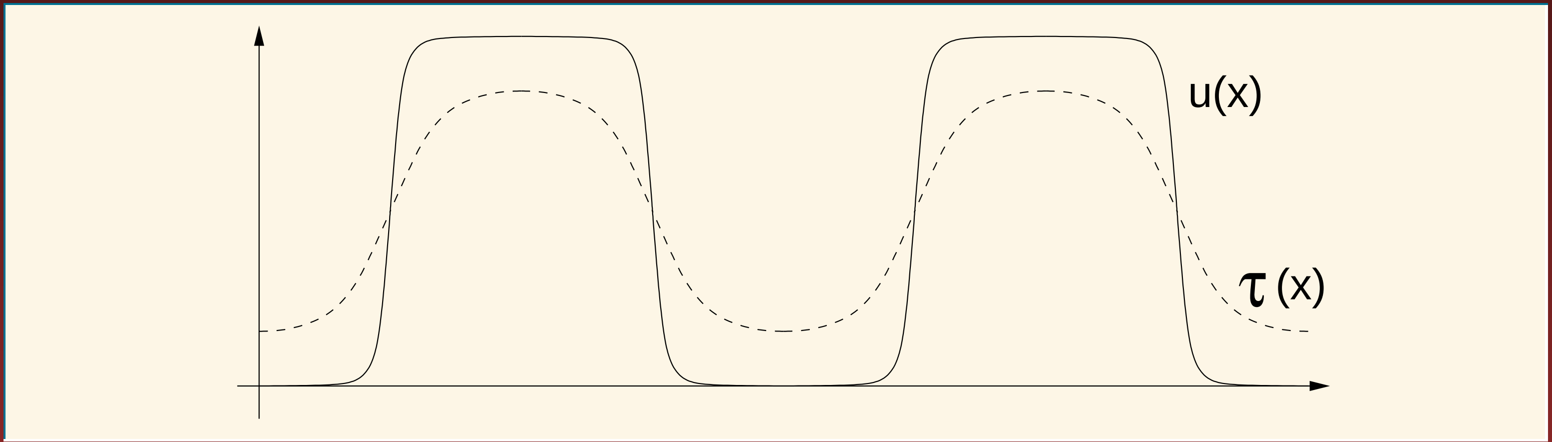
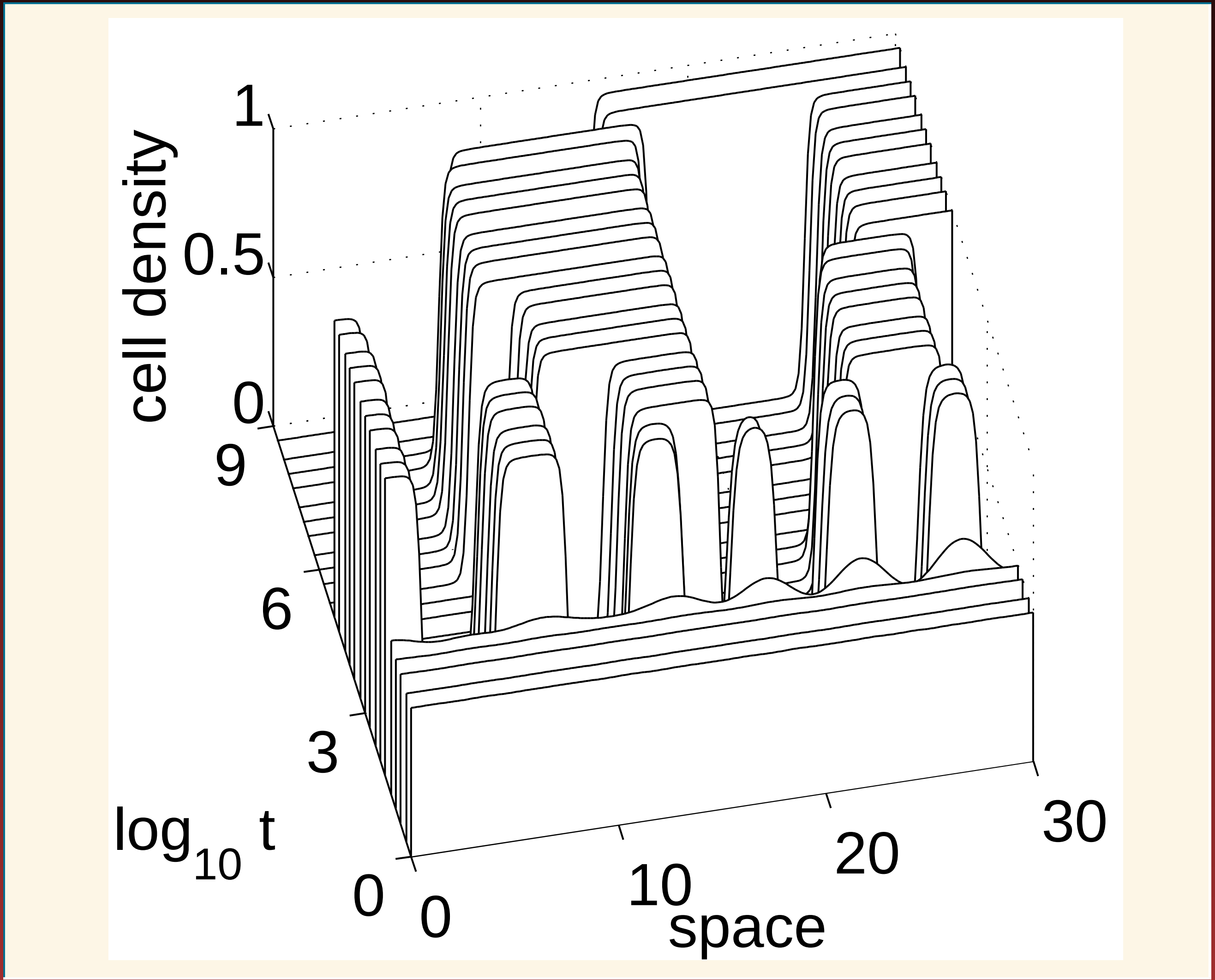
If the domain is large enough we obtain non trivial steady states.



Pattern Formation in 1-D

$f(u, v) = u - v.$



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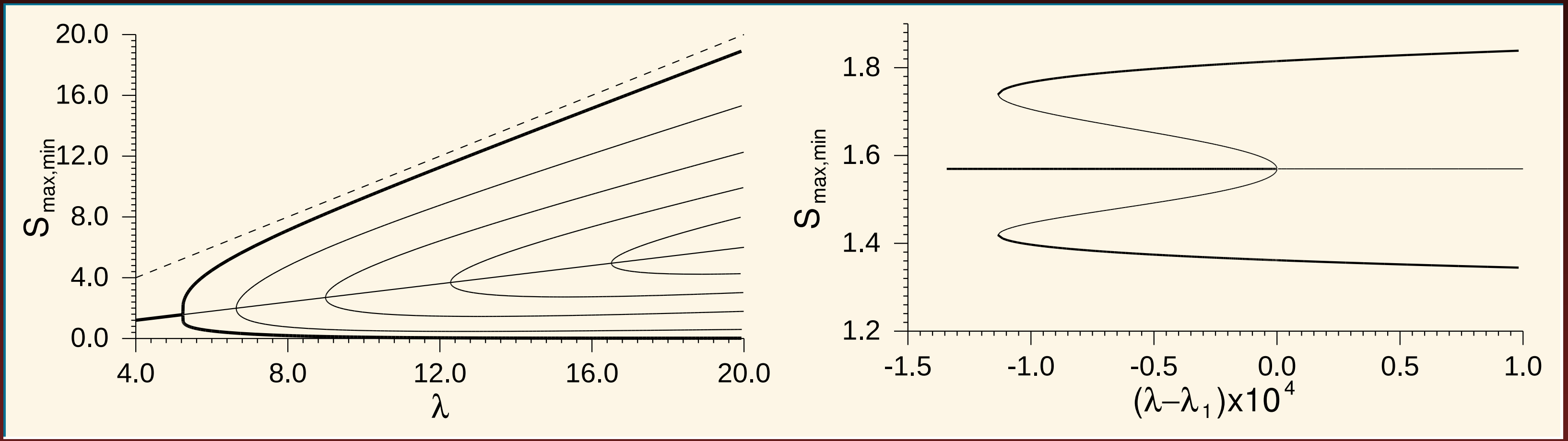
- [4] *D. Wrzosek 2004:*
  Lyapunov function. ω-limit sets are steady states.

- [2] *Hillen + Painter 2002:*
  Pattern formation, coarsening, interaction of spatial patterns on long time scales.

- [2] *Hillen + Painter 2002:*
  Pattern formation, coarsening, interaction of spatial patterns on long time scales.
- [5] *Potapov + Hillen 2004:*
  Bifurcation diagram, metastability, numnerical estimates of leading eigenvalues, scaling analysis and pattern interaction.

Bifurcation Diagram

(with Potapov)



- [6] *Dolak + Schmeiser 2004:* Asymptotic analysis of pattern interaction.

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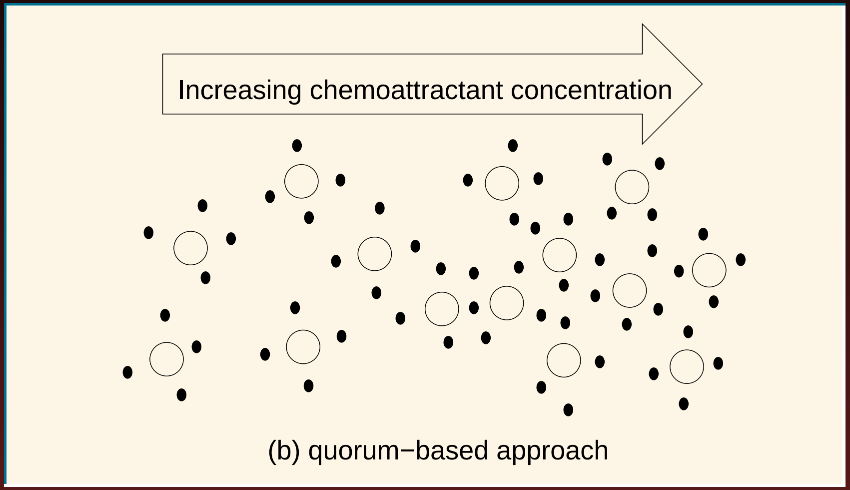
  Asymptotic analysis of pattern interaction.
- [7] *Dolak + Hillen 2003:*
  Application to *Dyctiostelium discoideum* and to *Salmonella typhimurium*.

Quorum Sensing



w: concentration of quorum sensing molecule.

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$\beta(w)$ describes the modified sensitivity. We assume β is decreasing in w.

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\begin{aligned}
u_t &= \nabla(D_u \nabla u - \chi u \beta(w) \nabla v) \\
v_t &= D_v \Delta v + f(u, v, w) \\
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Special case: If $\beta(w(u)) = 1 - u/U^*$ then a volume filling model follows.

Case 2: Non-Interfering Substances

$$
\begin{aligned}
u_t &= \nabla(D_u \nabla u - \chi_v u \nabla v + \chi_w u \nabla w) \\
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Painter, Maini, Othmer 2000: "Multiple cues".

The (v, w) dynamic is decoupled from the u dynamics. And it is assumed that (v, w) has a Turing instability.

Analysis of the non-intefering model

(w. J. Renclawowicz)

$$
\begin{aligned}
u_t &= \nabla(D_u \nabla u - \chi_v u \nabla v + \chi_w u \nabla w) \\
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Some rescaling:

$$
\begin{aligned}
u_t &= \nabla(\nabla u - u \nabla v + u \nabla w) \\
v_t &= D_v \Delta v + u - \beta v \\
w_t &= D_w \Delta w + \gamma u - \delta w
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Concentration difference, z

Introduce $z := v - w$:

$$
\begin{aligned}
u_t &= \Delta u - \nabla(u \nabla z) \\
z_t &= D_v \Delta v + D_w \Delta w + (1-\gamma)u - \beta v - \delta w
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$$\begin{aligned} u_t &= \Delta u - \nabla(u \nabla z) \\ z_t &= D \Delta z + (1-\gamma)u - \beta z \end{aligned}$$

Repulsive Case

If $\gamma > 1$ we obtain

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Or with $y = -z$:

$$\begin{aligned} u_t &= \Delta u + \nabla(u\nabla y) \\ y_t &= D\Delta y + u - \beta y \end{aligned}$$

Attractive Case

If $\gamma < 1$ we obtain

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which coincides with the classical chemotaxis model for $z \geq 0$!!

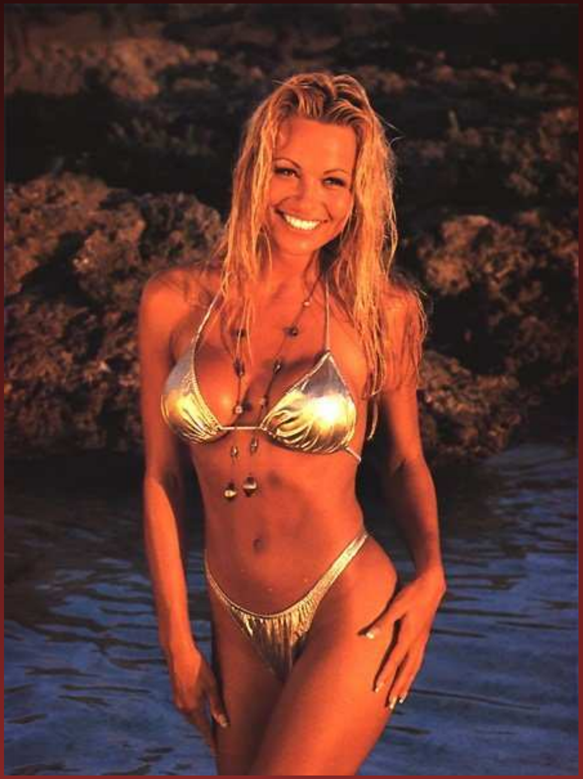
Super Model

$$
\begin{aligned}
u_t &= \Delta u + a\nabla(u\nabla z) \\
z_t &= D\Delta z + u - \beta z
\end{aligned}
$$

$$a = \pm 1$$

Super Model

Super Model



Results

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- Local existence for the super model.
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- In 1-D, global existence for the super-model.
  Norm-estimates of Hillen + Potapov 2004.
- 2-D: Lyapunov function for the aggregative case.
  Modify Gajewsky + Zacharias 1998 and Biler 1998 for $z <> 0$.
  Consequence: Global existence below a threshold $\int u_0 < 4D\theta$.

More Results

- 2-D: Existence of blow-up solutions.
  Herrero-Velazquez 1997

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- n-D, $n \geq 3$:
  Assume $p > n/2 + 1$ and $\|u_0\|_p, \|\nabla z_0\|_p$ are small. Then solutions of the super model exist globally in time.
  New proof based on L^p estimates.

(c) Finite Sampling Radius

(with Painter and Schmeiser)

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Othmer, H' (2002): Nonlocal gradient:

$$\mathring{\nabla}\, v(x,t) := \frac{n}{|S^{n-1}|R} \int_{S^{n-1}} \sigma v(x + R\sigma, t)\, d\sigma.$$

Sampling radius R.

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Sampling radius R.

(i) If v =const. then $\overset{\circ}{\nabla} v = 0$.

(ii) For each (x,t) we have

$$\lim_{R \to 0} \overset{\circ}{\nabla} v = \nabla v.$$

Modified Model

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\begin{aligned}
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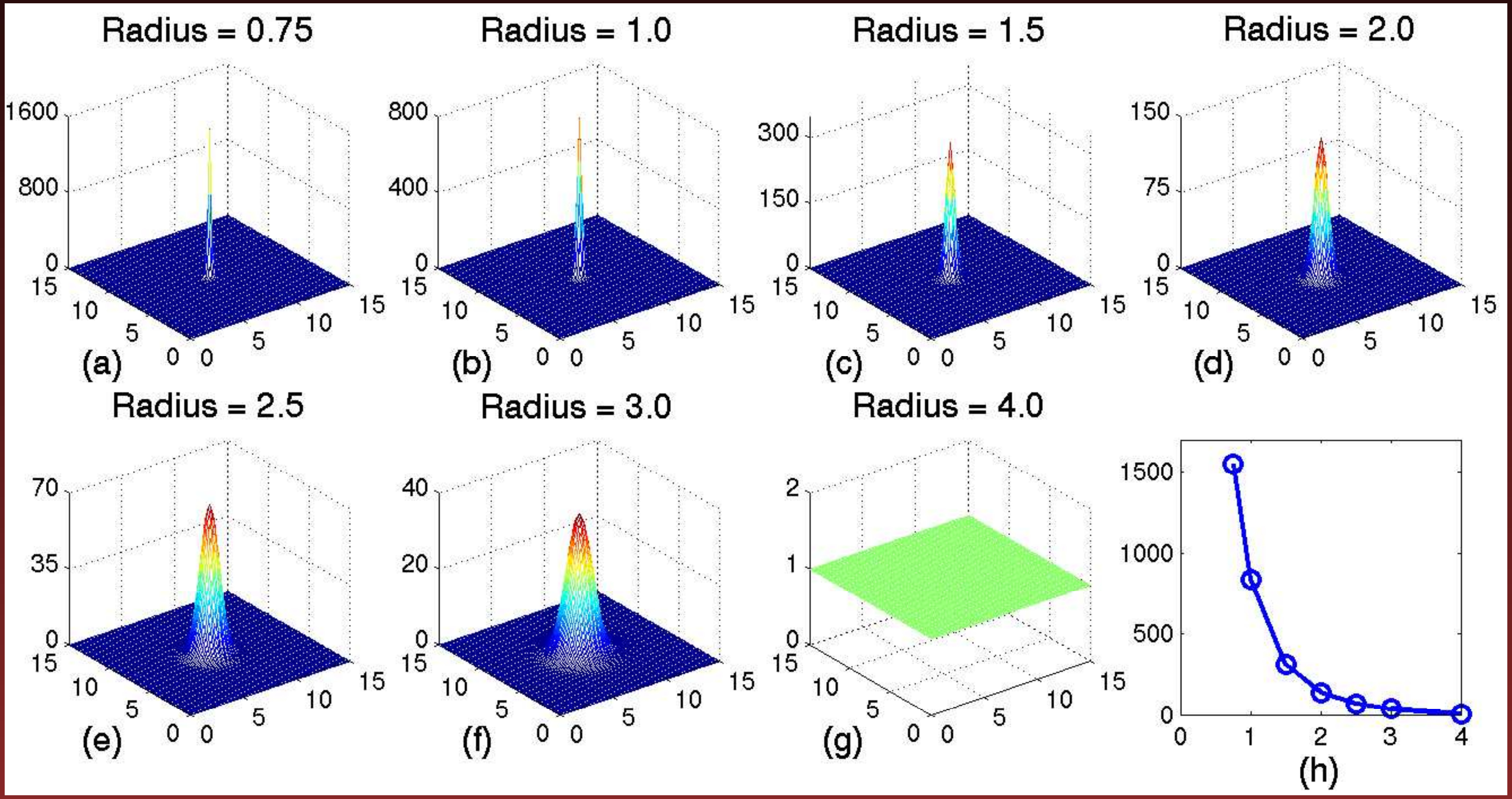
$$u_t = \Delta u - \chi \nabla \cdot \left\{ u \overset{\circ}{\nabla} v \right\}$$
$$v_t = D_v \Delta v + \gamma u - \delta v$$

Theorem :

$n = 2, R > 0 \Longrightarrow$ global existence.

Simulations



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Globally existing solutions and interesting pattern interaction.

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Open question: Does the non local gradient have additional regularity properties?