General relative entropy inequality and structured population models

Benoît Perthame[1], Stéphane Mischler[2] and Philippe Michel [1,2]

November 7, 2004

[1] Ecole Normale Supérieure, DMA, UMR 8553,
45, rue d'Ulm, F75230 Paris cedex 05
perthame@dma.ens.fr, jabin@dma.ens.fr

[2] CEREMADE, Univ. Paris Dauphine
Pl. de Lattre de Tassigny, F75775 Paris cedex 16
mischler@ceremade.dauphine.fr

1-PBE models (population balance equation) We study the dynamic of a cell population describes by its density at time t, $n(t,y)$, in which the intracellular state is characterized by y (size, age or ...).

2-Examples of size structured models

$$\frac{\partial}{\partial_t} n(t,y) + \frac{\partial}{\partial_y} n(t,y) + B(y)n(t,y) = 4B(2y)n(t,2y),$$
$$n(t,0) = 0, \quad n(0,y) = n_0(y),$$

or more generally

$$\frac{\partial}{\partial_t} n(t,y) + \frac{\partial}{\partial_y} n(t,y) + B(y)n(t,y) = \int_y^\infty b(y,y')n(t,y')dy',$$
$$n(t,0) = 0, \quad n(0,y) = n_0(y).$$

3-Example of age structured model

$$\frac{\partial}{\partial_t} n(t,a) + \frac{\partial}{\partial_a} n(t,a) + D(y)n(t,y) = 0,$$
$$n(t,0) = \int B(y)n(t,y)dy, \quad n(0,y) = n_0(y).$$

I-Aim

Our purpose is to show that in these cases, we have the following asymptotic behaviour such that $n(t,y) \sim e^{\lambda t} N(y)$ as $t \longrightarrow \infty$ for a

$$\lambda \in \mathbb{R},$$
$$N \in L^1(\mathbb{R}_+), \quad N(y) \geq 0, \quad N \not\equiv 0.$$

A scattering model :

$$\partial_t f = \int f(y')K(y,y')dy' - \int f(y)K(y',y)dy',$$

then there is mass conservation. Now, if we have the existence of N solution to $N(y')K(y,y') = N(y)K(y',y)$, then for all H positive convex function

$$\partial_t \int H(f/N)N(y)dy = \int\int \left[H'(f/N)(y)(\frac{f}{N}(y') - \frac{f}{N}(y)) \right.$$
$$\left. +(H(f/N)(y) - H(f/N)(y'))\right] K(y,y')N(y')dydy' \leq 0,$$

(where $H'(z) = d/dz H(z)$) and for some 'good' K we have $f \longrightarrow N$.

Formal conservative law :

Let F a linear operator (cell division...)

$$\frac{\partial}{\partial_t} n(t,y) = F(n).$$

formally we have the following conservative law

$$\frac{\partial}{\partial_t} \int n(t,y)\phi(y)dy = \int F(n)(y)\phi(y)dy = \int n(t,y)F^*(\phi)(y)dy = 0$$

as $F^*(\phi) = 0$, $\phi > 0$.

A cell division model (with constant rate of division):

$$\frac{\partial}{\partial_t} n(t,y) + \frac{\partial}{\partial_y} n(t,y) + Bn(t,y) = 4Bn(t,2y),$$

then $\partial_t \int n(t,y)dy = B \int n(t,y)dy$ and we normalize n by

$m = ne^{-Bt}$ thus

$$\frac{\partial}{\partial_t} m(t,y) = F(m) = -\frac{\partial}{\partial_y} m(t,y) - 2Bm(t,y) + 4Bm(t,2y),$$

$$F^*(\phi) = \frac{\partial}{\partial_y}\phi(t,y) - 2B\phi(t,y) + 2B\phi(t,y/2),$$

we have clearly $F^*(1) = 0$ and $\int m(t,y)1dy = \int m(0,y)1dy$.

II- Entropy

General entropy equation for these non-conservative equations $\partial_t n = F(n)$ (size or age). Let $m = ne^{-\lambda t}$ then for all H convex, positive function, we have

$$\partial_t \int H(m/N)N(y)\phi(y)dy = -D_H(m/N) \leq 0,$$

where N, ϕ are defined by the eigenproblem

$$F(N) = -\lambda N, \quad \int N(y)dy = 1, \quad N(y) \geq 0,$$

$$F^*(\phi) = -\lambda \phi, \quad \int N(y)\phi(y)dy < \infty, \quad \phi > 0,$$

$$\partial_t m = F(m) - \lambda m.$$

Entropy decreasing term D_H :

For all H convex, positive function, we have

In the size structured model

$$\partial_t \int H(m/N)N(y)\phi(y)dy =$$

$$-\int\int N(y')\phi(y)b(y,y')\big[H'(\frac{m}{N})(y)(\frac{m}{N}(y)-\frac{m}{N}(y'))+H(\frac{m}{N})(y')-H(\frac{m}{N})(y)\big]dy'dy$$

In the age structured model

$$\partial_t \int H(m/N)N(y)\phi(y)dy = H(\int m/N(y)d\mu(y)) - \int H(m/N)(y)d\mu(y),$$

$$d\mu(y) = b(y)N(y)dy/\int b(y')N(y')dy'.$$

We only use algebraic calculation and equations satisfied by N, ϕ.

III- Results

Under following assumptions

1- Existence of (λ, N, ϕ) solution to the eigenproblem

$$F(N) = -\lambda N, \quad \int N(y)dy = 1, \quad N(y) \geq 0,$$
$$F^*(\phi) = -\lambda \phi, \quad \int N(y)\phi(y)dy < \infty, \quad \phi > 0.$$

2- Some assumptions on $b(.,.)$, $B(.)$ $(B(.), D(.))$ that give $D_H(f) = 0$ iff f is constant,

we have

$$n(t,.)e^{-\lambda t} \to CN(y) \quad in \;\; L^p(\mathbb{R}_+, N(y)\phi(y)dy).$$

Moreover we have $\int n(t,y)e^{-\lambda t}\phi(y)dy = \int n(0,y)\phi(y)dy$.

IV- Link with evolution

We obtain $n(t,y) \sim e^{\lambda t} N(y)$, thus the λ is a measure of adaptation. For example let

$n_1(t,y)$ solution to the size struct. with $b(y,y') = \delta_{y=y'/2} + \delta_{y=y'/2}$,

$n_2(t,y)$ solution to the size struct. with $b(y,y') = \delta_{y=y'/3} + \delta_{y=2y'/3}$,

then $n_1 \sim N_1 e^{\lambda_1 t}$ and $n_2 \sim N_2 e^{\lambda_2 t}$. In this case we have $\lambda_1 = \lambda_2 = 1$ and the two species evolve with the same exponential growth.

In general, the exact value of λ is impossible to find and the link between parameters $(b,\ B)$ and λ is not clear. Nevertheless one can find numerically the solution of the eigenproblem.

V- Summary

1-Advantage

a-Entropy equation We can compute $\partial_t \int N\phi H(n/N)dy$ independently of the existence of N, ϕ-algebraic calculation.

b-Existence N, ϕ, λ In discrete and compact cases we have the existence of the solution to the eigenproblem.

c-Result $n \sim e^{\lambda t} N(y)$.

2-Disadvantage

a-Speed rate (for convergence) Hidden in $D_H(.)$,

b-λ No accurate information in general.

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