MATHEMATICS OF KELLER-SEGEL SYSTEM

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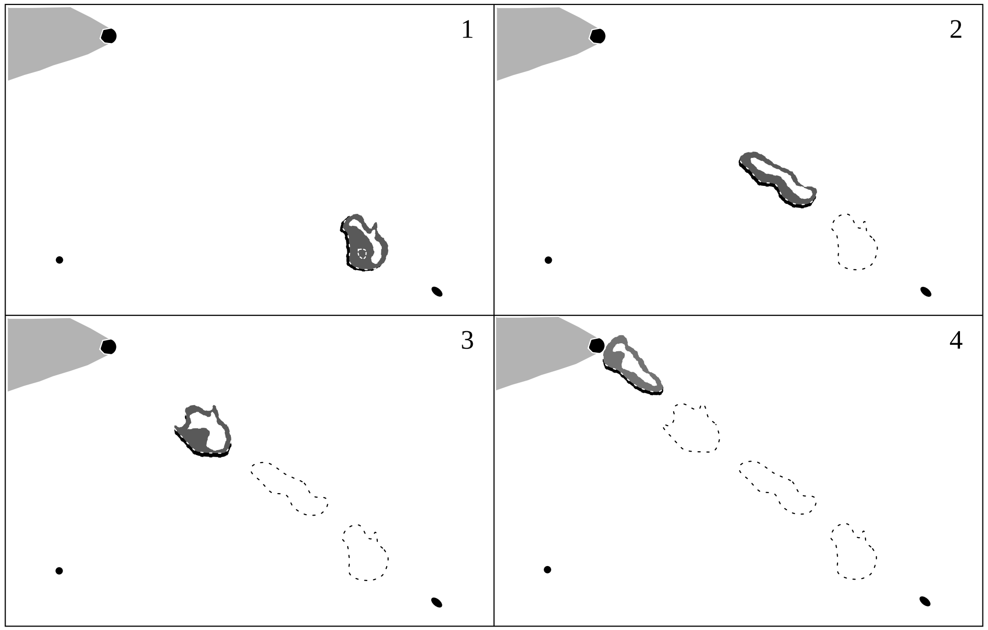

CHEMOTAXIS : mathematical models

The mathematical modelling of cell movement goes back to Patlak (1953), E. Keller and L. Segel (70's)

$$\begin{aligned} n(t,x) &= \text{ density of cells at time t and position } x, \\ c(t,x) &= \text{concentration of chemoattractant}, \end{aligned}$$

In a collective motion, the chemoattractant is emited by the cells that react according to biased random walk.

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) - \Delta n(t,x) + \text{div}(n \chi \nabla c) = 0, \quad x \in R^d, \\ -\Delta c(t,x) = n(t,x), \end{cases}$$

The parameter χ is the sensitivity of cells to the chemoattractant.

CHEMOTAXIS : mathematical models

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) - \Delta n(t,x) + \text{div}(n\chi\nabla c) = 0, & x \in R^d, \\ -\Delta c(t,x) = n(t,x), \end{cases}$$

This model, although very simple, exhibits a deep mathematical structure an dmostly only dimension 2 is understood, especially "chemotactic collapse".

This is the reason why it has attracted a number of authors.

CHEMOTAXIS : mathematical models

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) - \Delta n(t,x) + \text{div}(n\chi\nabla c) = 0, \quad x \in R^d, \\ \\ -\Delta c(t,x) = n(t,x), \\ \\ n(t=0) = n^0(x) \geq 0, \quad n^0 \in L^1(R^d). \end{cases}$$

-Childress, Parcus (84) ; Jäger, Luckhaus (92),
-Rascle, Zitti (95) ; Nagai (95) ; Biler, Nadzieja(93),
-Herrero, Medina, Velazquez (96-03) ;
-Brenner, Constantin, Kadanoff, Schenkel, Venkatarami (98) ;
-Horstmann (00) ; Corrias, Dolbeault, Perthame, Zaag (04) ;

OUTLINE OF THE LECTURE

I. General existence/blow-up

II. Case of dimension 2

III. Numerical blow-up

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CHEMOTAXIS : general result

Theorem (dimensions $d \geq 2$)

(i) for $\|n^0\|_{L^{d/2}(R^d)}$ small, then there are global weak solutions,

(ii) these small solutions propagate L^p regularity,

(iii) for $(\int |x|^2 n^0)^{(d-2)} < C \|n^0\|^d_{L^{d/2}(R^d)}$ with C small, there is blow-up in a finite time T^*.

CHEMOTAXIS : general result

The existence proof relies on Jäger-Luckhaus argument

$$\frac{d}{dt}\int n(t,x)^p = -\frac{4}{p}\int |\nabla\, n^{p/2}|^2 + \underbrace{\int p\nabla n^{p-1}\, n\, \chi\, \nabla c}_{\chi\int \nabla n^p\cdot\nabla c = -\chi\int n\Delta c}$$

$$= \underbrace{-\frac{4}{p}\int |\nabla n^{p/2}|^2}_{\text{parabolic dissipation}} + \underbrace{\chi\int n^{p+1}}_{\text{hyperbolic effect}}$$

Using Gagliardo-Nirenberg-Sobolev ineq. on the quantity $u(x) = n^{p/2}$, we obtain

$$\int n^{p+1} \leq C_{\text{gns}}(d,p)\|\nabla n^{p/2}\|^2_{L^2}\; \|n\|_{L^{\frac{d}{2}}}$$

CHEMOTAXIS : general result

Therefore we arrive at

$$\frac{d}{dt}\int n(t,x)^p \le -\frac{4}{p}\int |\nabla n^{p/2}|^2 \Big(1 - C_{\text{gns}}\, \chi\, \|n\|_{L^{\frac{d}{2}}}\Big).$$

And the choice $p = d/2$ shows an a priori bound for $\|n\|_{L^{\frac{d}{2}}}$ initially small (because it decreases then).

CHEMOTAXIS : dimension 2

In dimension 2, for Keller and Segel model :

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) - \Delta n(t,x) + \mathrm{div}(n\chi\nabla c) = 0, \quad x \in R^2, \\ -\Delta c(t,x) = n(t,x), \end{cases}$$

Theorem (dimension d=2) Dolbeault, Perthame
(i) for $\|n^0\|_{L^1(R^2)} < \frac{8\pi}{\chi}$, then there are smooth solutions,
(iii) for $\|n^0\|_{L^1(R^2)} > \frac{8\pi}{\chi}$ and $\int |x|^2 n^0 < \infty$ there is blow-up time.

This is sharp and improves J. L. result : their thereshold

$$\|n^0\|_{L^1(R^2)} < \frac{4\pi \times 1.822}{\chi}$$

CHEMOTAXIS : dimension 2

Existence part is based on an energy method

$$\frac{d}{dt}\left[\int_{R^2} n \log n \, dx - \frac{\chi}{2}\int_{R^2} n\, c\, dx\right] = -\int_{R^2} n \left|\nabla (\log n) - \chi \nabla c\right|^2 dx \, .$$

and a limiting Hardy-Littlewood-Sobolev inequality (Carlen-Loos)

$$\int_{R^2} f \log f \, dx + \frac{2}{M} \int \int_{R^2 \times R^2} f(x) f(y) \log|x-y| \, dx\, dy \geq M(1 + \log \pi + \log M) \, .$$

Notice that in $d = 2$ we have

$$c(t,x) = \frac{1}{2\pi} \int n(t,y) \log|x-y| \, dy$$

CHEMOTAXIS : dimension 2

Combining these two informations, we deduce that

$$\begin{aligned}\int n(x,t) \log n(x,t)\, dx &\le \mathcal{E}_0 - \frac{\chi}{4\pi} \int\int n(x,t)\, n(y,t)\, \log|x-y|\, dx\, dy \\ &\le \mathcal{E}_0 - \frac{\chi}{4\pi} C(M) + \frac{M\chi}{8\pi} \int n(x,t) \log n(x,t)\, dx\ .\end{aligned}$$

And again we obtain a priori control.

CHEMOTAXIS : dimension 2

On the other hand blow-up follows from the idendity

$$\frac{d}{dt} \int_{R^2} |x|^2 \, n(x,t) \, dx = 4M \left(1 - \frac{\chi}{8\pi} M \right).$$

Very old method (Zhakharov, Glassey, Nagai).

Conclusion

Several open questions are

-) Is it possible to prove blow-up for

$$\|n^0\|_{L^{d/2}} \quad \text{large (without moment condition),}$$

-) Despite the knowledge of possible blow-up modalities, proofs are not rigorous,

-) More elaborate systems (see V. Calvez).