DECAYING TURBULENCE: THEORY AND EXPERIMENTS

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Richardson-Kolmogorov cascade

Kolmogorov (1941): $E(k) \approx C_K \epsilon^{2/3} k^{-5/3}$
for $1/L \ll k \ll 1/\eta$
assuming ϵ, the K.E. dissipation rate per unit mass, to be (G.I. Taylor 1935)

$$\epsilon = C_\epsilon u'^3/L$$

with C_K and C_ϵ indep of Re_λ for $Re_\lambda \gg 100$

Note: $L/\eta \sim Re^{3/4}$, i.e. $L/\lambda \sim Re_\lambda$
Richardson-Kolmogorov cascade

LES modelling

Eddy viscosities in LES from Richardson-Kolmogorov cascade

$$\nu_t = \frac{2}{3} C_K^{-3/2} \epsilon^{1/3} k_c^{-4/3}$$

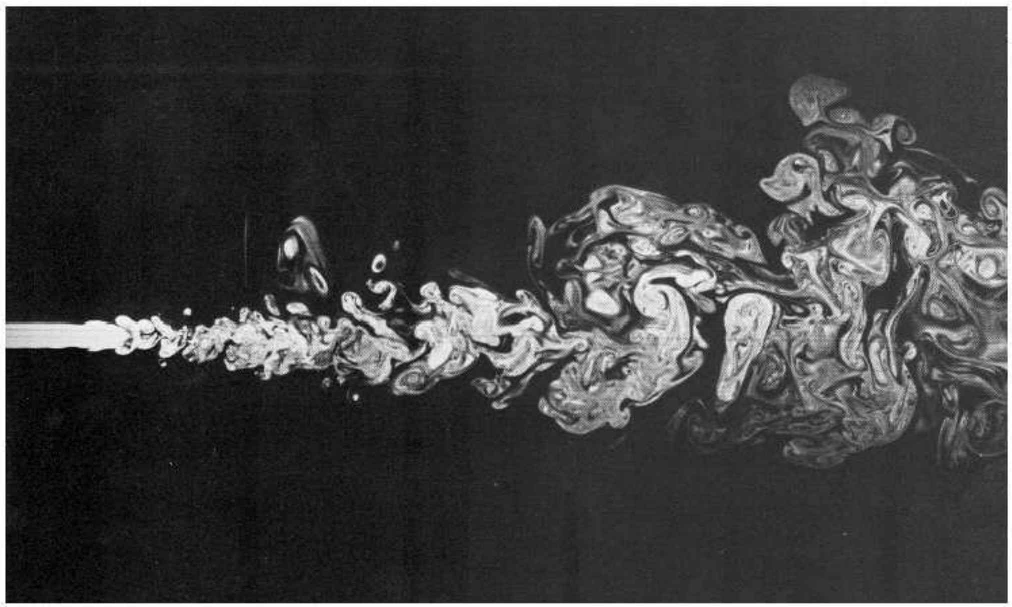
where $\epsilon = C_\epsilon u'^3 / L$ with C_ϵ indep of Re_λ

$$\nu_t = \frac{2}{3} C_K^{-3/2} C_\epsilon^{1/3} u' L (k_c L)^{-4/3}$$

with universal values of C_K and C_ϵ.

Turbulent jets

Jets

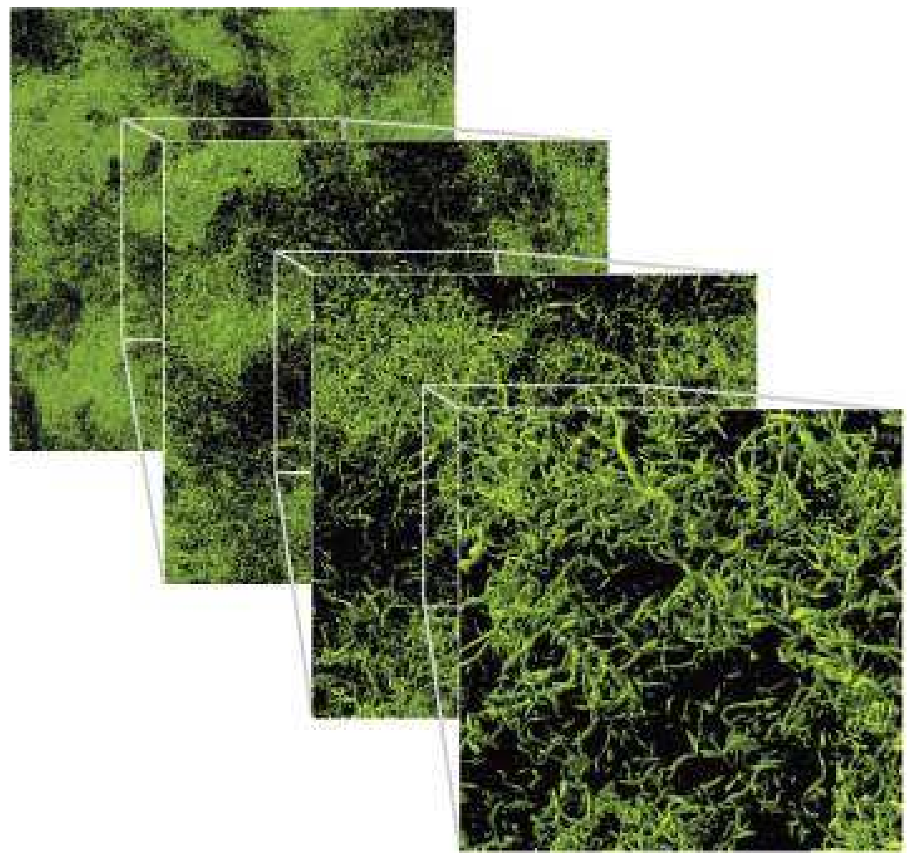


Turbulent wakes

FIG. 1.

Homogeneous turbulence

(from Ishihara et al, early/mid 2000s, Japan, Earth Simulator calculations)



Some background

	WT HT	*Axisym Jet*	*Plane Jet*
u'	$U_\infty(\frac{x-x_0}{L_b})^{-p}$, $1/2 < p < 3/4$	$U_\infty(\frac{x-x_0}{L_b})^{-1}$	$U_\infty(\frac{x-x_0}{L_b})^{-1/2}$
L_u	$L_b(\frac{x-x_0}{L_b})^{q}$, $0 < q < 1/2$	$x - x_0$	$x - x_0$

	Axisym Wake	*Plane Wake*	*Mixing Layer*
u'	$U_\infty(\frac{x-x_0}{L_b})^{-2/3}$	$U_\infty(\frac{x-x_0}{L_b})^{-1/2}$	U_∞
L_u	$L_b^{2/3}(x - x_0)^{1/3}$	$L_b^{1/2}(x - x_0)^{1/2}$	$(x - x_0)$

L_b: characteristic cross-stream length-scale of inlet (e.g. mesh or nozzle or bluff body size...)
U_∞: characteristic inlet mean flow velocity or mean flow velocity cross-stream variation

Some background

	WT HT	Axisym Jet	Plane Jet
L_u/λ	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{-p}$	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{0}$	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{1/4}$
Re_λ	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{-p}$	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{0}$	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{1/4}$

	Axisym Wake	Plane Wake	Mixing Layer
L_u/λ	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{-1/6}$	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{0}$	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{1/2}$
Re_λ	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{-1/6}$	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{0}$	$Re_0^{1/2}(\frac{x-x_0}{L_b})^{1/2}$

$L_u/\lambda \sim Re_\lambda$ in all cases

λ obtained from $\epsilon \sim u'^3/L_u \sim \nu u'^2/\lambda^2$

$Re_0 \equiv \frac{U_\infty L_b}{\nu}$: inlet Reynolds number

Decaying H(I)T

Sedov (1944, 1982) and George (1992) found exact single-scale (self-preserving) solutions of Lin's equation $\frac{\partial E(k,t)}{\partial t} = T(k,t) - 2\nu k^2 E(k,t)$.

E.G. It admits exact solutions of the form
$E(k,t) = u'^2(t)l(t)e(kl(t), i.c./b.c.)$
and
$T(k,t) = \frac{d}{dt}[u'^2(t)l(t)]\tau(kl(t), i.c./b.c.)$.
(See George 1992; George & Wang 2009.)

These solutions are such that $L \sim l(t)$ and $\lambda \sim l(t)$; hence L/λ remains constant during decay even though Re_λ can decay fast. This implies $L/\lambda \sim Re_\lambda^0$ to be contrasted with the Richardson-Kolmogorov cascade's $L/\lambda \sim Re_\lambda$.

Decay of two-scale cascading H(I)T

The decay of homogeneous isotropic cascading turbulence is traditionally considered to obey the following constraints:

1. $\frac{d}{dt}\frac{3}{2}u'^2 = -\epsilon$ where $\epsilon \sim u'^3/L$ –equivalent to $L/\lambda \sim Re_\lambda$.

2. Invariants of the von Kárman-Howarth equation (physical space equivalent of the Lin equation):

(i) Either the Loitsyansky invariant $u'^2 \int_0^{+\infty} r^4 f(r) dr$ is non-zero and Const in time;

(ii) Or the Birkhoff-Saffman invariant $u'^2 \int_0^{+\infty} \left[3r^2 f(r) + r^3 \frac{\partial f(r)}{\partial r}\right] dr$ is non-zero and Const in time.

3. $u'^2 f(r,t) \equiv < u(x,t)u(x+r,t) >$ is self-similar at large enough r, i.e. $f(r,t) \approx f[r/L(t)]$ if r not too small.

Invariants of von Kárman-Howarth

$$\frac{\partial}{\partial t}(u'^2 f) = u'^3(\frac{\partial k}{\partial r} + \frac{4k}{r}) + 2\nu u'^2(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r})$$

where $u'^3 k(r,t) \equiv < u^2(x,t)u(x+r,t) >$,

$$I_{Mnn'} \equiv u'^2 \int_0^{+\infty} r^{M+n'} \frac{\partial^{n'} f(r)}{\partial r^{n'}} dr + C_{Mnn'} u'^2 \int_0^{+\infty} r^{M+n} \frac{\partial^n f(r)}{\partial r^n} dr$$

are all invariants of the von Kárman-Howarth equation provided that $M > 1$, $\lim_{r\to\infty}(r^M k) = 0$ and $\lim_{r\to\infty}(r^{M-1} f) = 0$ and that $I_{Mnn'}$ is well-defined. $M = 4$ is the Loitsyansky and $M = 2$ is the Birkhoff-Saffman inv.

The von Kárman-Howarth equation admits an infinity of possible finite integral invariants depending on conditions at infinity.

Consequences of these invariants

When $M > 1$ and $M \neq 4$,
assuming
(i) $f(r,t) \approx a_{M+1}(t)(L(t)/r)^{M+1}$ to leading order when $r \to \infty$
and
(ii) $\lim_{r\to\infty}(r^M k) = 0$,

then I_{Mn0} is finite for all $n \geq 1$ and its invariance leads to

$$\frac{d}{dt}(a_{M+1}L^{M+1}u'^2) = 0.$$

This proves a precise version of the principle of permanence of large eddies.

None or one or two invariants

For conditions at infinity such that the Birkhoff-Saffman invariant is not infinite, either none or only one or only two invariants are finite.

Assuming that there exists a number $M_f \geq 2$ for which $\lim_{r\to\infty}(r^{M_f+1}f) = a_{M_f+1}L^{M_f+1} \not\equiv 0$ and a number M_g for which $\lim_{r\to\infty}(r^M k) = 0$ for any M in the interval $2 \leq M < M_g$ but $\lim_{r\to\infty}(r^M k) \neq 0$ for any $M \geq M_g$, then we have the following four possibilities.

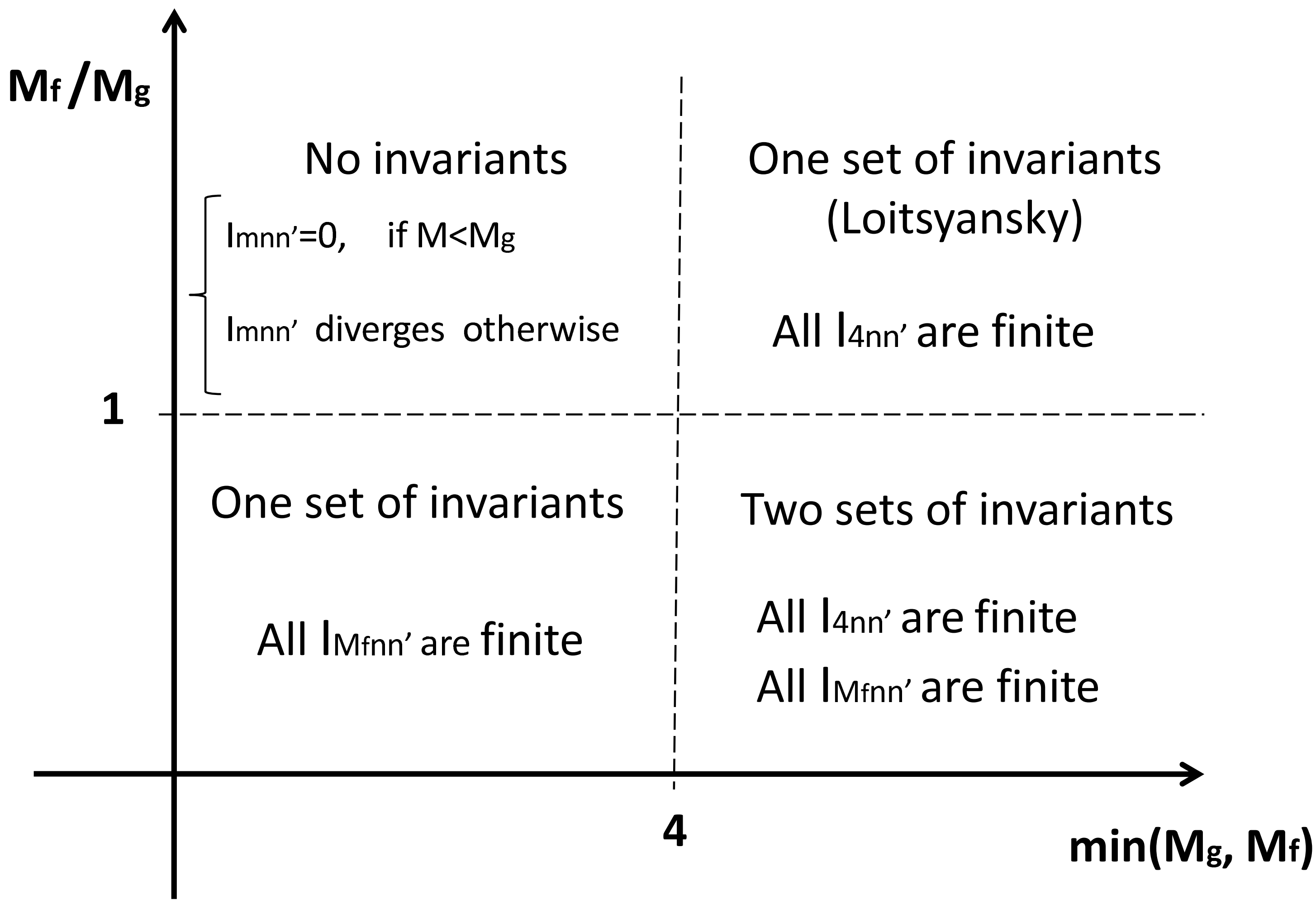
(1) $M_f/M_g > 1$, $min(M_f, M_g) < 4$: no finite invariants

(2) $M_f/M_g > 1$, $min(M_f, M_g) \geq 4$: Loitsyansky invariant

(3) $M_f/M_g \leq 1$, $min(M_f, M_g) < 4$: $\frac{d}{dt}(a_{M_f+1}L^{M_f+1}u'^2) = 0$

(4) $M_f/M_g \leq 1$, $min(M_f, M_g) > 4$: $\frac{d}{dt}(a_{M_f+1}L^{M_f+1}u'^2) = 0$ and Loitsyansky. If $min(M_f, M_g) = 4$ only Loitsyansky.

None or one or two invariants



Implications for self-preserving decay

George (1992) exact single-scale solutions of the von Kármann-Howarth equation:

$f(r,t) = f[r/l(t)]$ and $k(r,t) = b(\nu, u_0', l_0, t-t_0)\kappa[r/l(t)]$

Solvability conditions ($\alpha > 0$, $c > 0$):

$u'^2(t) = {u_0'}^2 \left[1 + \frac{c\nu}{l_0^2}(t-t_0)\right]^{-2\alpha/c}$ and $l^2(t) = l_0^2 + c\nu(t-t_0)$

(1) $M_f/M_g > 1$, $min(M_f, M_g) < 4$: any α and c.

(2) $M_f/M_g > 1$, $min(M_f, M_g) \geq 4$: $2\alpha/c = 5/2$

(3) $M_f/M_g \leq 1$, $min(M_f, M_g) < 4$: $2\alpha/c = (M_f+1)/2$ and lies between 3/2 and 5/2 as $2 \leq M_f < 4$

(4) $M_f/M_g \leq 1$, $min(M_f, M_g) > 4$: self-preserving George solutions impossible. If $M_f = 4$ then $2\alpha/c = 5/2$

Implications for cascading decay

Decay of homogeneous isotropic cascading turbulence:

(i) $\frac{d}{dt}\frac{3}{2}u'^2 = -\epsilon$ where $\epsilon \sim u'^3/L$ –equivalent to $L/\lambda \sim Re_\lambda$.

(ii) $f(r,t) \approx f[r/L(t)]$ if r not too small.

(iii) Implications of von Kárman-Howarth invariants:

(1) $M_f/M_g > 1$, $min(M_f, M_g) < 4$: open.

(2) $M_f/M_g > 1$, $min(M_f, M_g) \geq 4$:

$u'^2(t) = {u'_0}^2\left[1 + c(t-t_0)\right]^{-10/7}$ and

$L(t) = L_0^2\left[1 + c(t-t_0)\right]^{-2/7}$

(3) $M_f/M_g \leq 1$, $min(M_f, M_g) < 4$:

$u'^2(t) = {u'_0}^2\left[1 + c(t-t_0)\right]^{-n}$ where $n = 2(M_f+1)/(M_f+3)$ lies between $6/5$ and $10/7$ as $2 \leq M_f < 4$.

$L(t) = L_0\left[1 + c(t-t_0)\right]^{-2/(M_f+3)}$.

(4) $M_f/M_g \leq 1$, $min(M_f, M_g) > 4$: large-scale self-similarity impossible. If $M_f = 4$ then (2).

Summary

Decay of two-scale (inner and outer) homogeneous isotropic cascading turbulence:

$$u'^2(t) = {u_0'}^2 \left[1 + c(t - t_0)\right]^{-n}$$

where n lies between $6/5 = 1.2$ and $10/7 = 1.43$.

Decay of self-preserving single-scale homogeneous isotropic turbulence:

$$u'^2(t) = {u_0'}^2 \left[1 + \frac{c\nu}{l_0^2}(t - t_0)\right]^{-2\alpha/c}$$

where $2\alpha/c$ lies between $3/2 = 1.5$ and $5/2 = 2.5$.

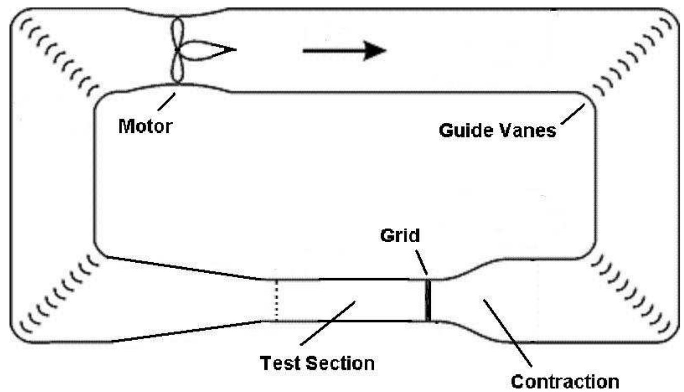
Summary

There exist asymptotic behaviours at infinity of the double and triple velocity correlation functions which are a priori possible and for which no finite invariant of the von Kármán-Howarth equation exists. In this case, it is unknown what sets the exponents n and $2\alpha/c$.

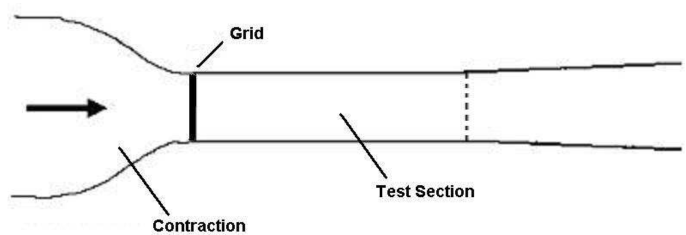
There exist asymptotic behaviours at infinity of the double and triple velocity correlation functions which are a priori possible and for which no self-preserving and no large-scale self-similar decays of homogeneous isotropic turbulence are possible.

Wind tunnels

$0.91^2 m^2$ width; test section $4.8m$; max speed $45m/s$; background turbulence ≈ 0.25%.



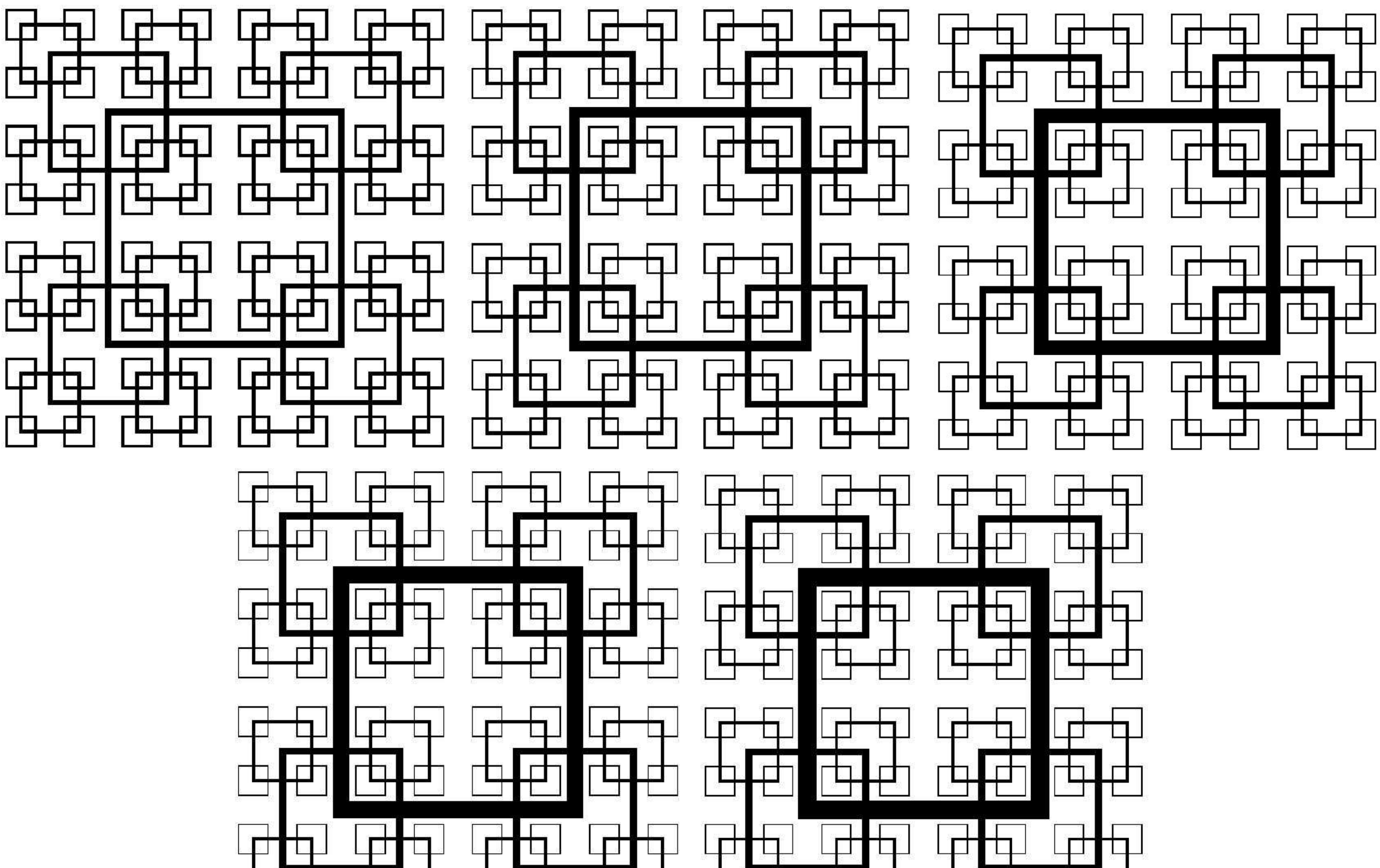
$0.46^2 m^2$ width; test section $\approx 4.0m$; max speed $33m/s$; background turbulence ≈ 0.4%.



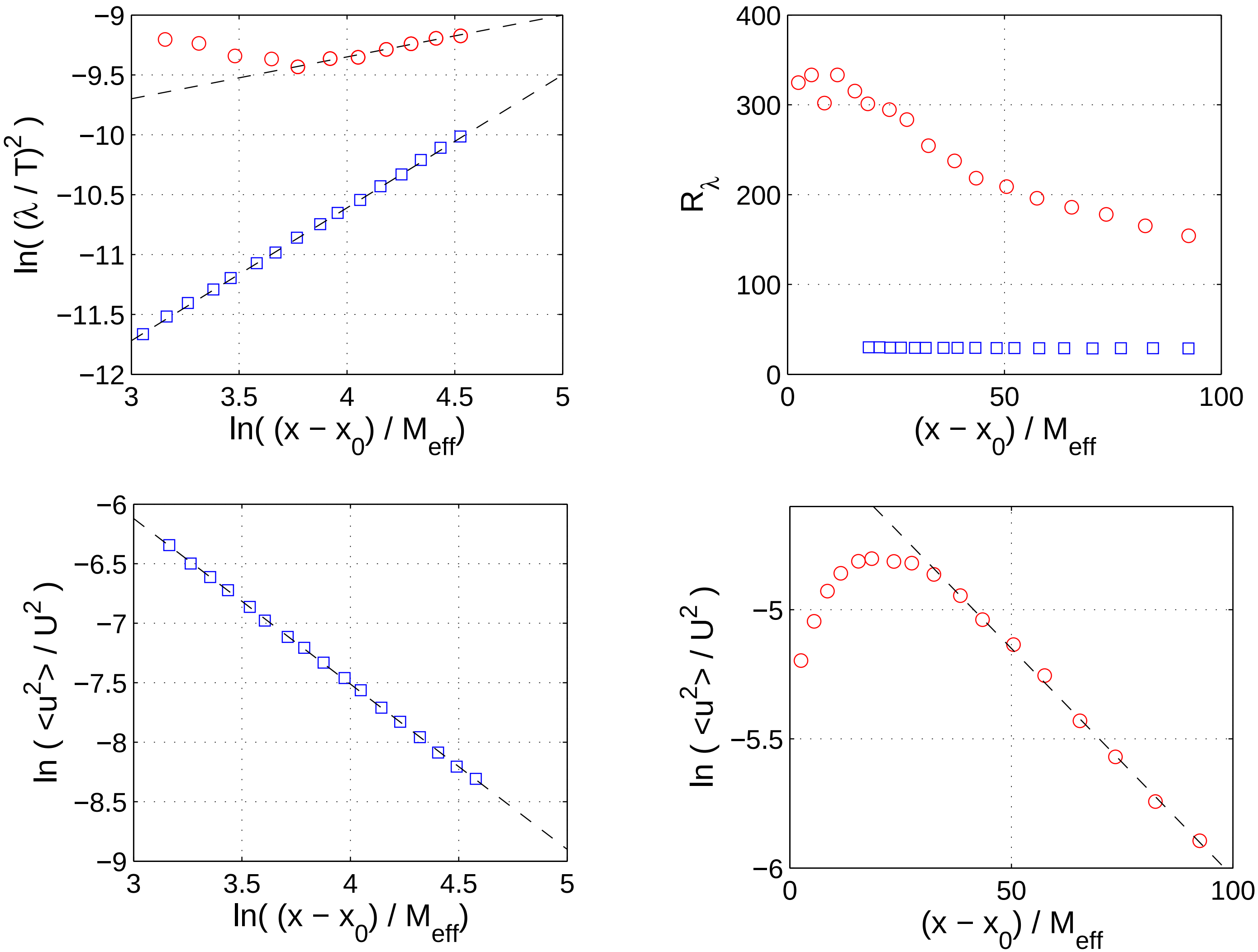
$D_f = 2$, $\sigma = 25$% fractal square grids

and equal $M_{eff} \approx 2.6cm$, $L_{max} \approx 24cm$, $L_{min} \approx 3cm$, $N = 4$, $T = 0.46m$.

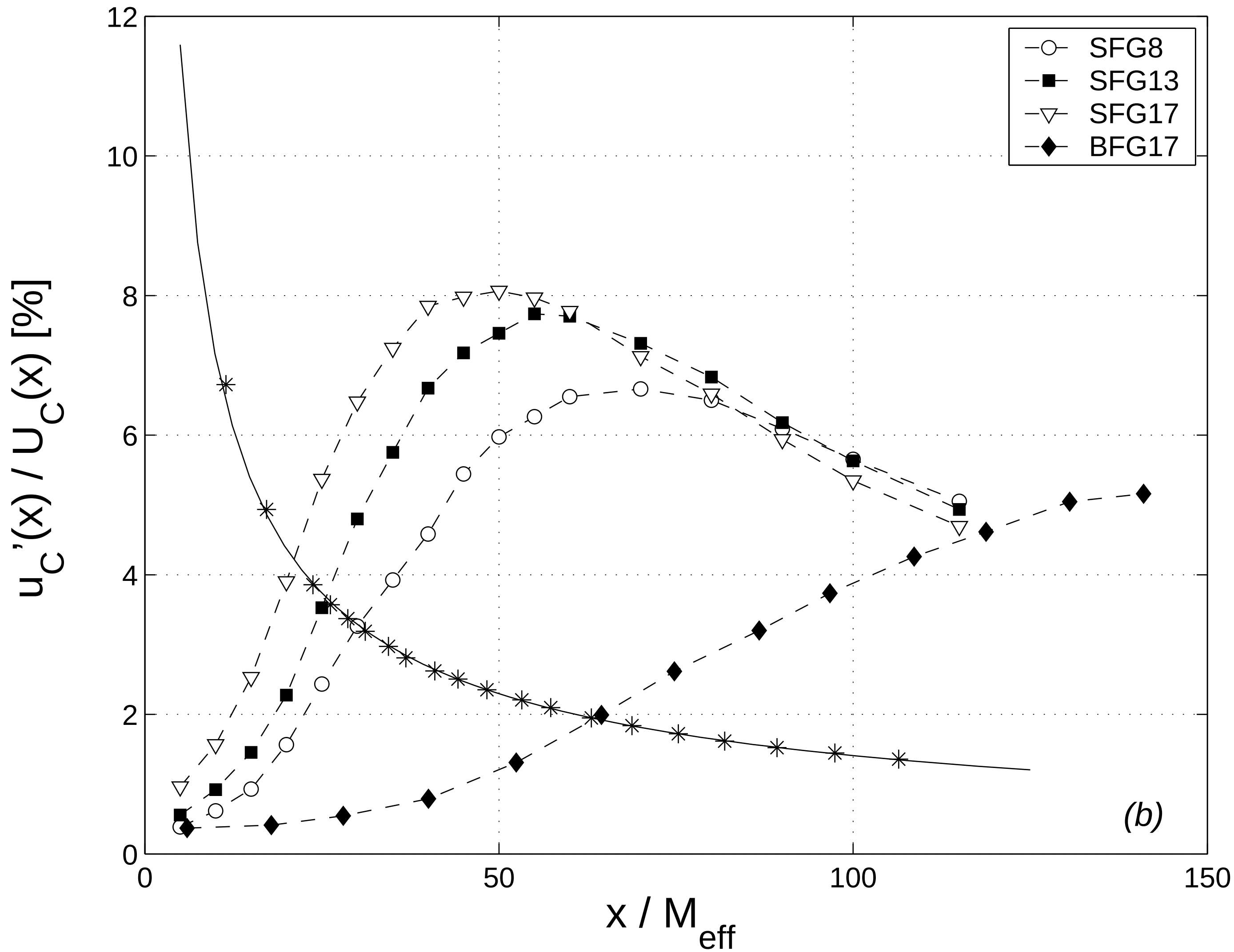
BUT $t_r = 2.5, 5.0, 8.5, 13.0, 17.0$



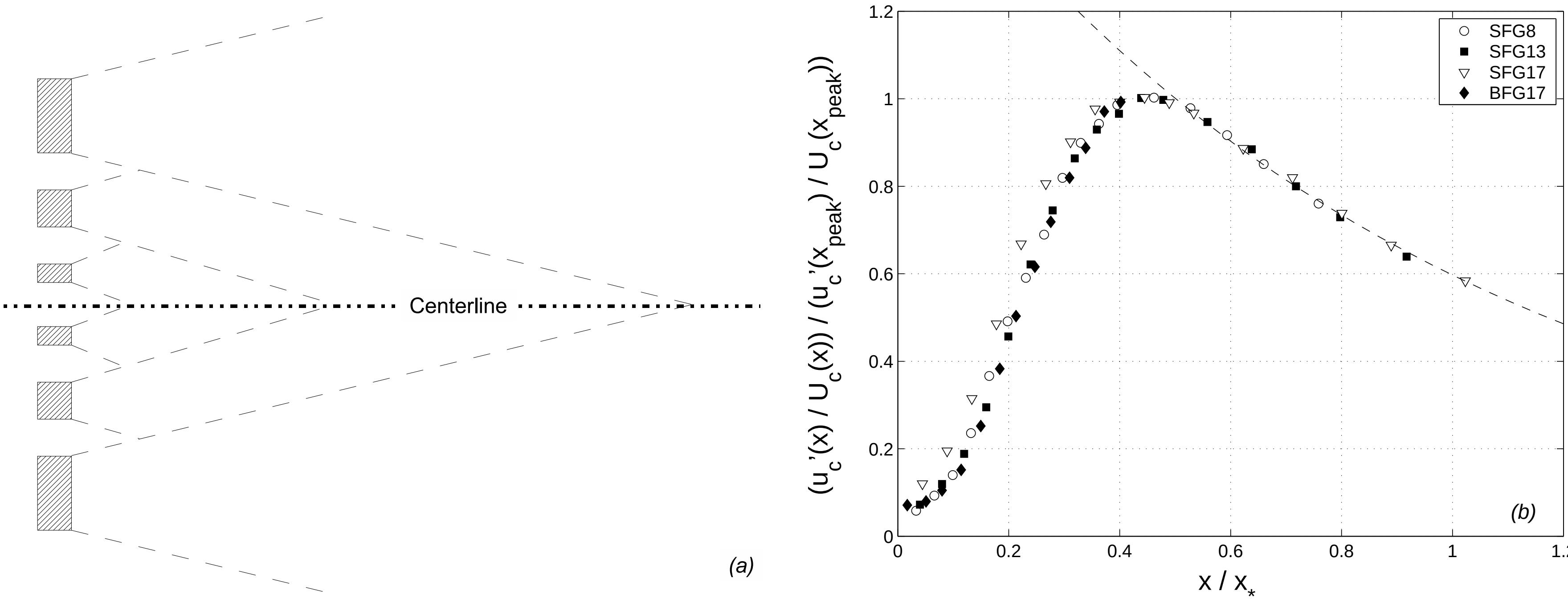
Comparison with regular grid turbulence



Streamwise turbulence intensity



Wake-interaction length-scale

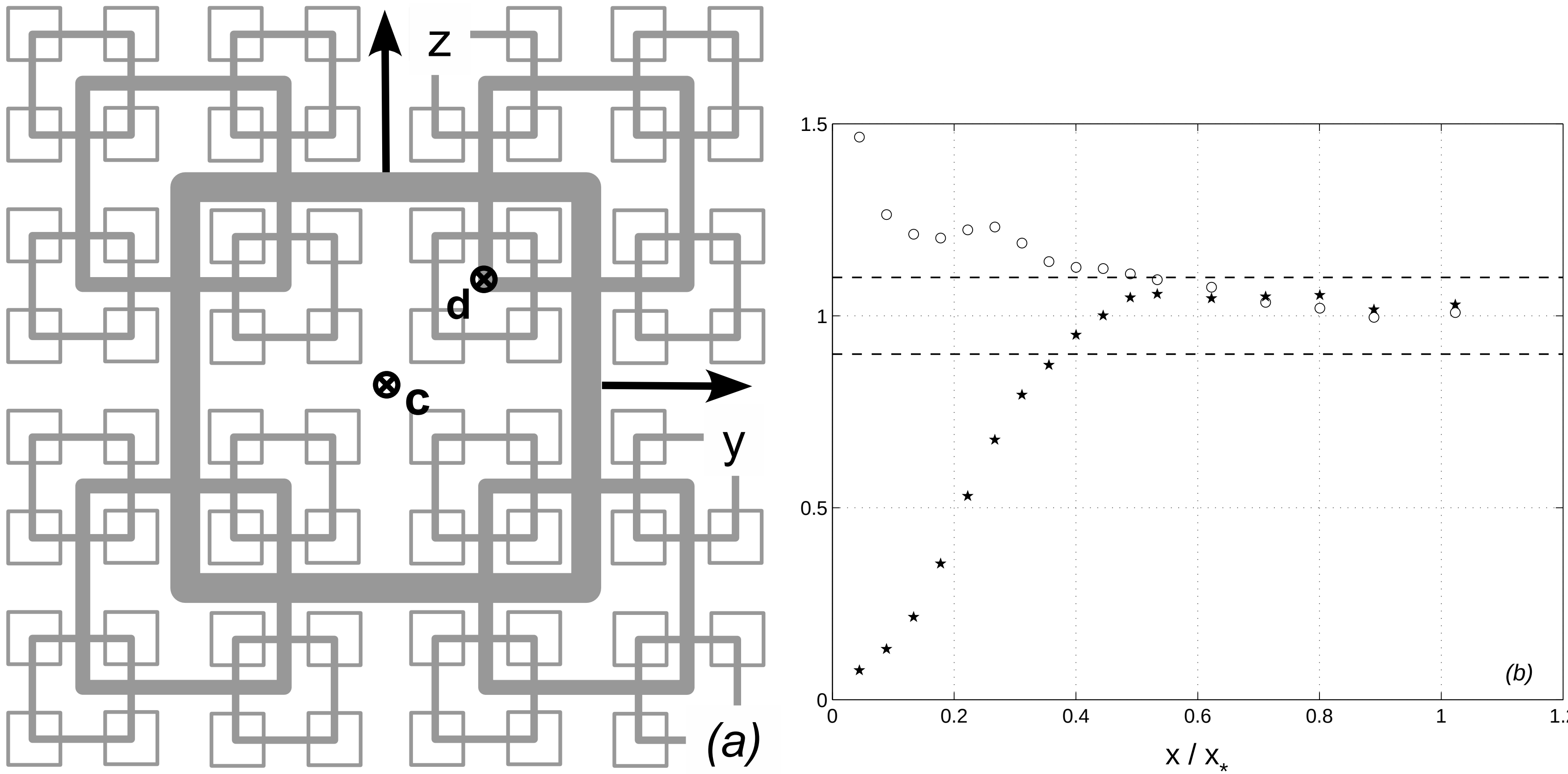


$(u'/U)/(u'/U)_{peak}$ versus x/x_* where $L_0 = \sqrt{t_0 x_*}$

$$x_{peak} \approx 0.5 x_*$$

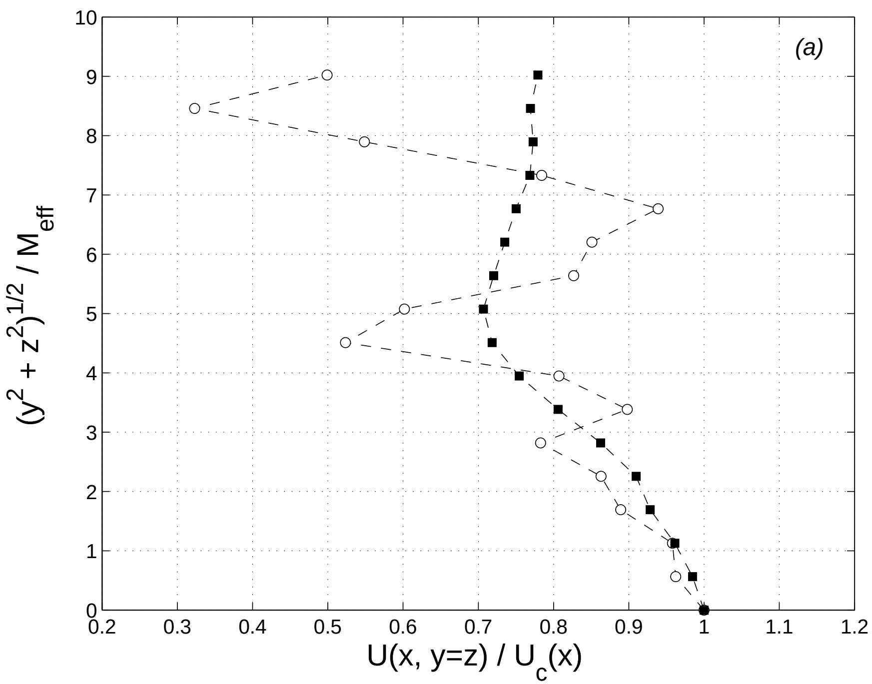
$u'/U \sim \exp(-Bx/x_*)$ where $x > x_{peak}$; $B \approx 2.06$.
In agreement with Seoud & V **19**, 105108 (2007).

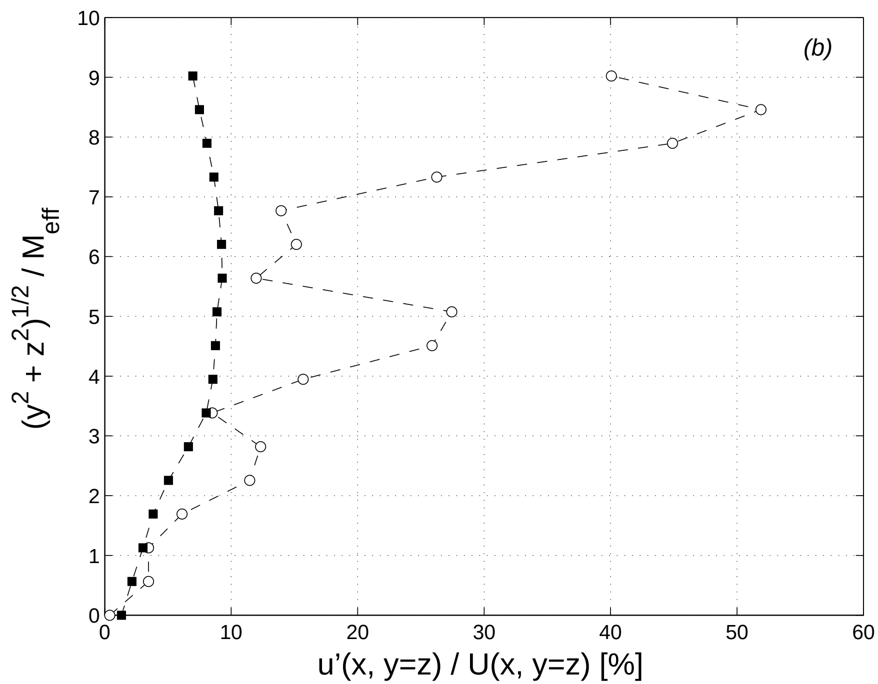
Homogeneity where $x > x_{peak}$



U_c/U_d and u'_c/u'_d versus x/x_*

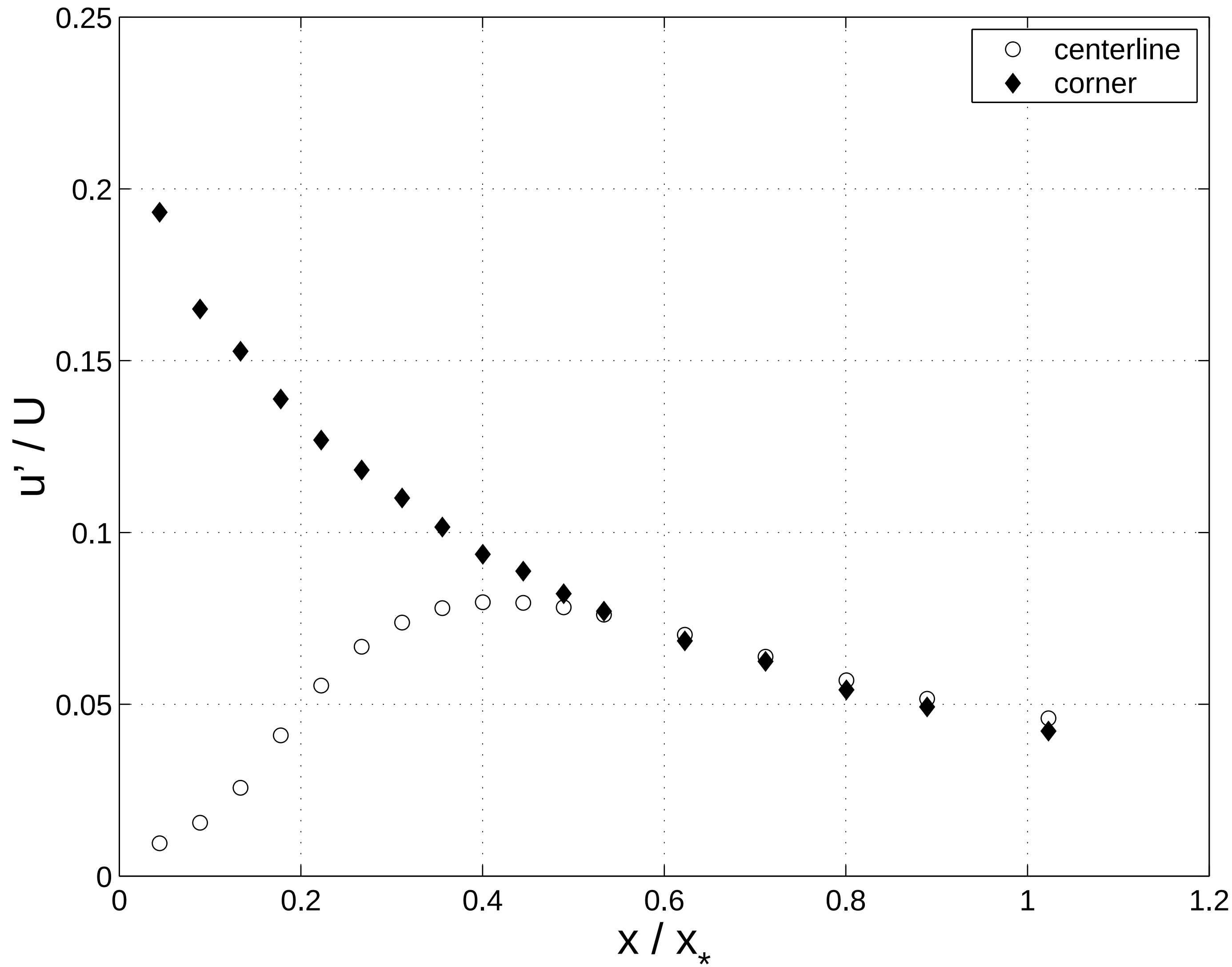
From inhomogeneity to homogeneity



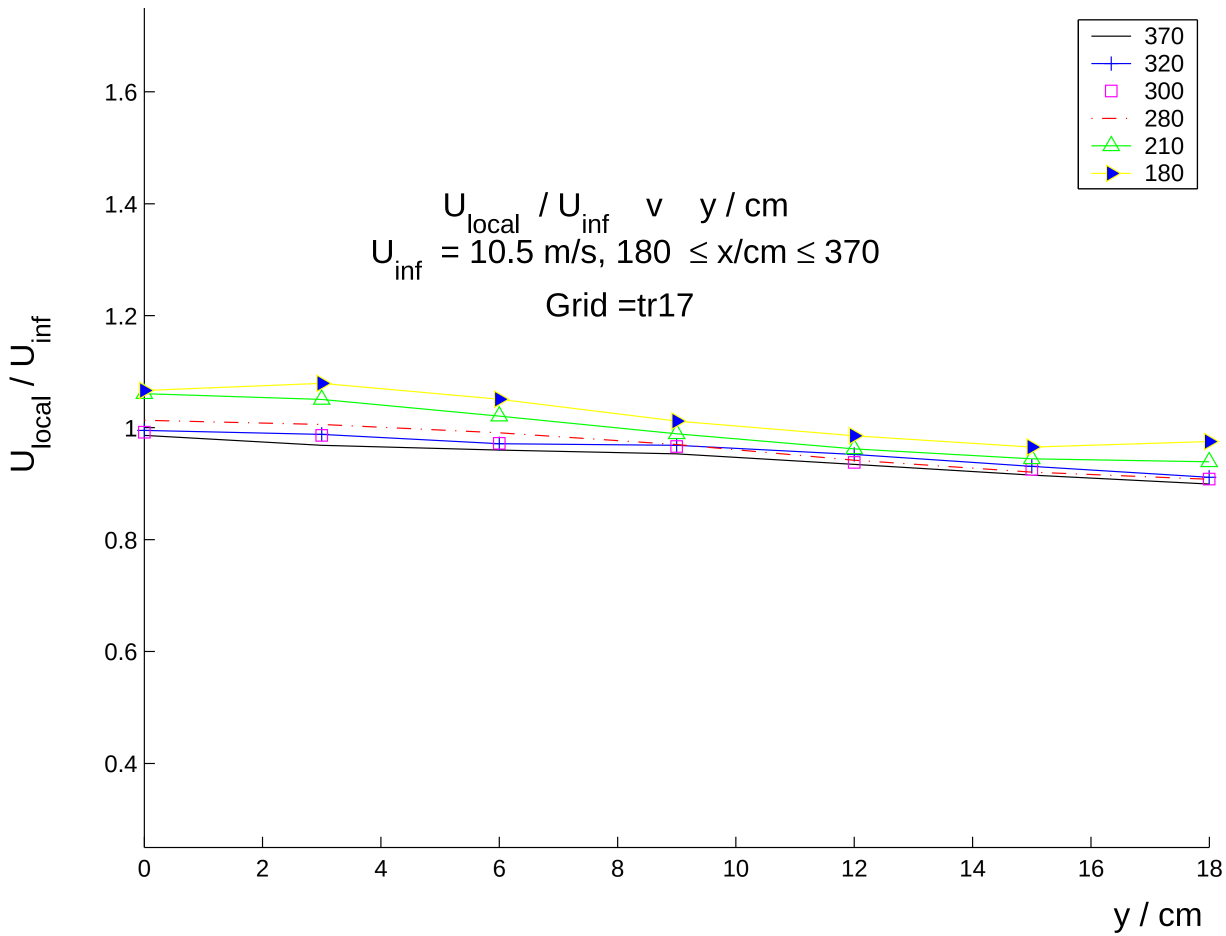


Mean flow and turbulence intensity profiles
at $x \approx 0.02x_*$ ($x \approx 7M_{eff}$) and $x \approx 0.15x_*$ ($x \approx 53M_{eff}$).

From inhomogeneity to homogeneity

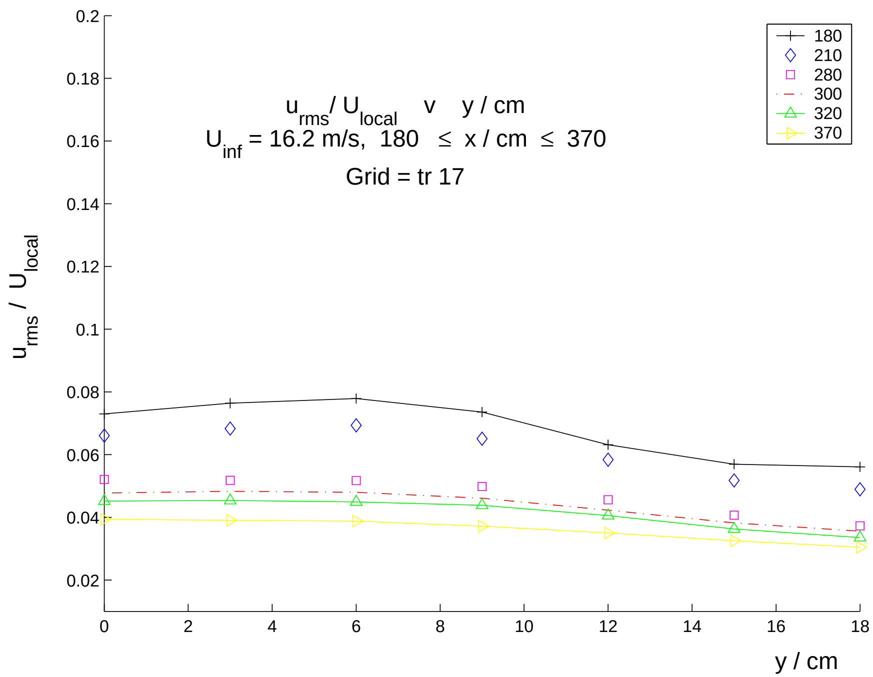


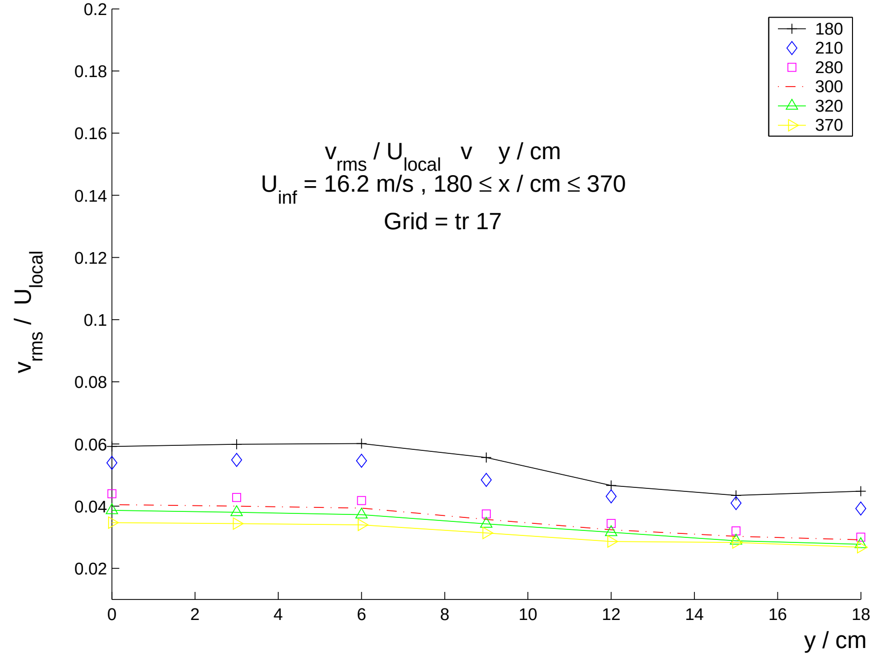
Statistical homogeneity at $x > x_{peak}$



From Seoud & V PoF, 2007.

Statistical homogeneity at $x > x_{peak}$

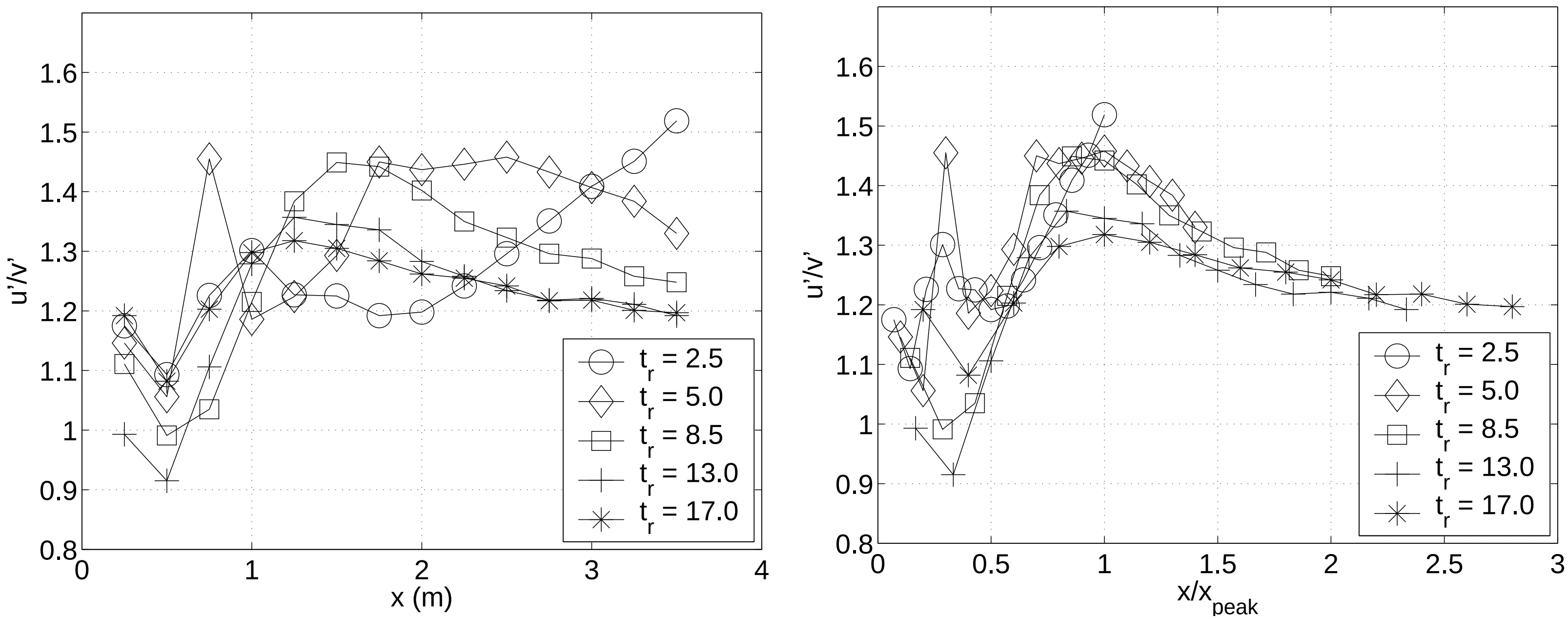




From Seoud & V PoF, 2007.

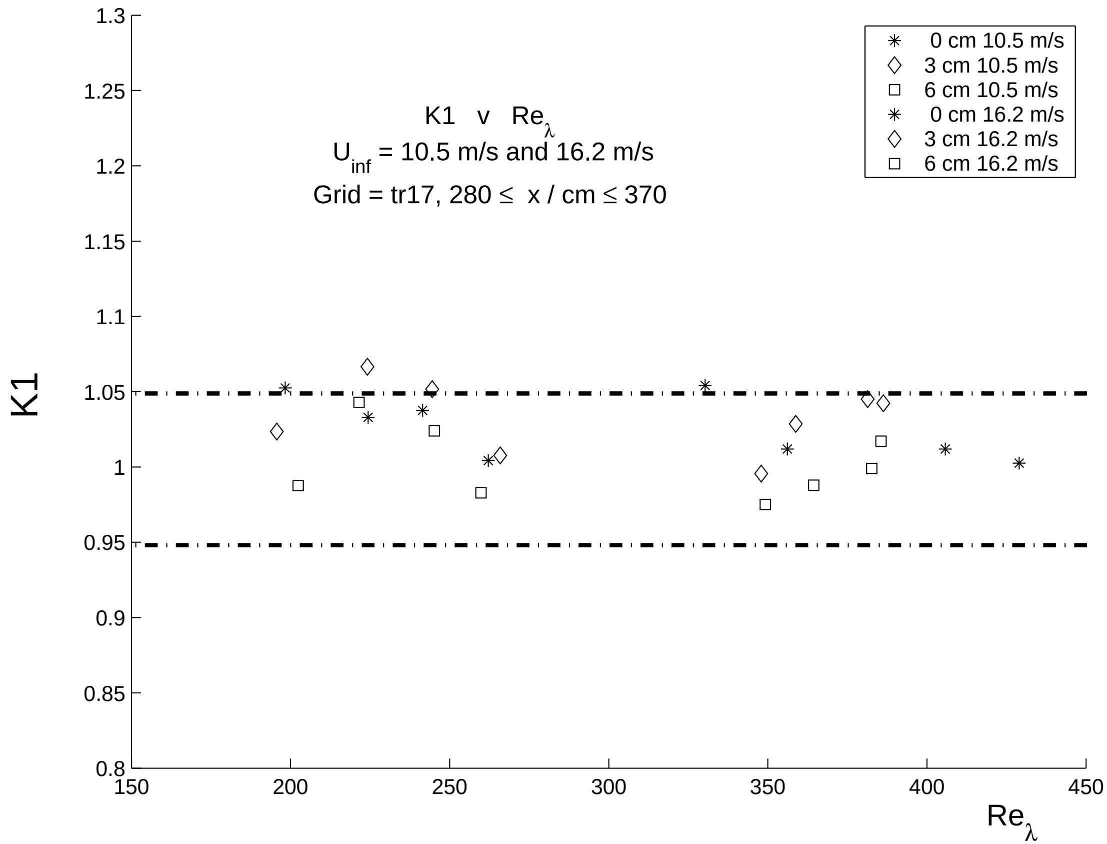
From non-isotropy to near-isotropy

x_{peak} helps collapse u'/v' as fct of x



From Hurst & V PoF, 2007; $T = 0.46m$ tunnel with U_∞ = 10m/s.

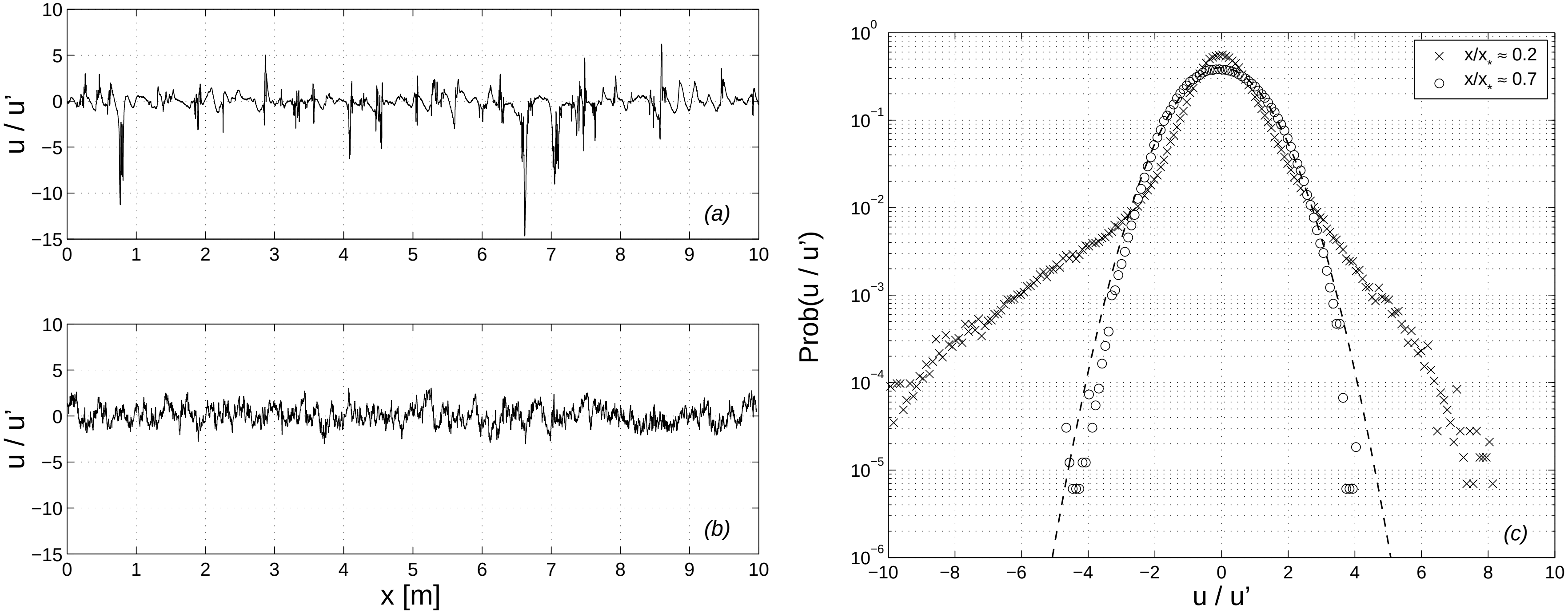
Statistical local isotropy at $x > x_{peak}$



From Seoud & V PoF, 2007: $K_1 \equiv 2 < (\frac{\partial u}{\partial x})^2 > / < (\frac{\partial v}{\partial x})^2 >$ as function of Re_λ at locations (x, y) downstream from $t_r = 17$ fractal grid where $x \geq 2x_{peak}$ and $y = 0, 3, 6cm$.

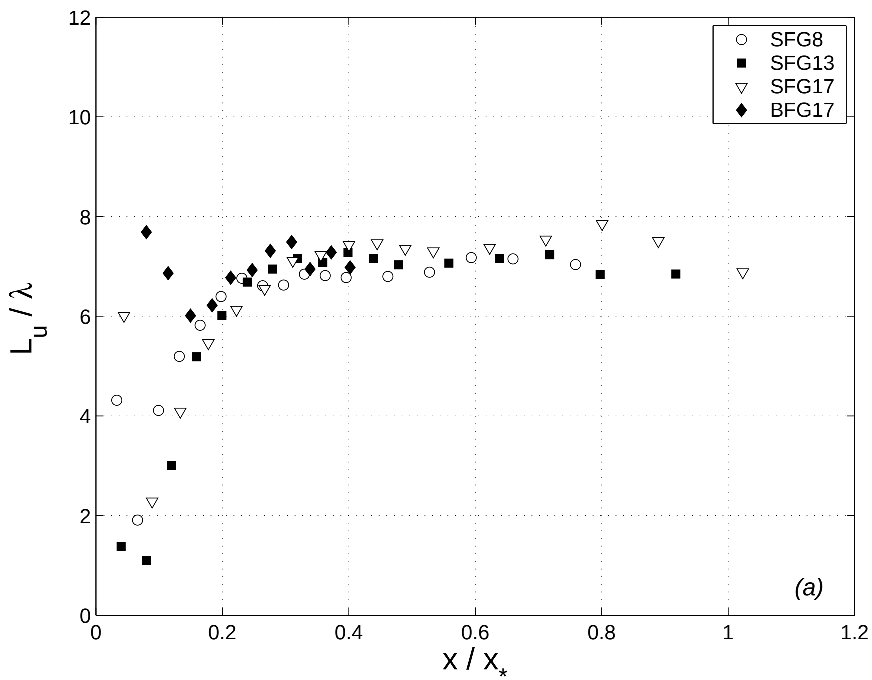
Local isotropy implies $K_1 = 1$.

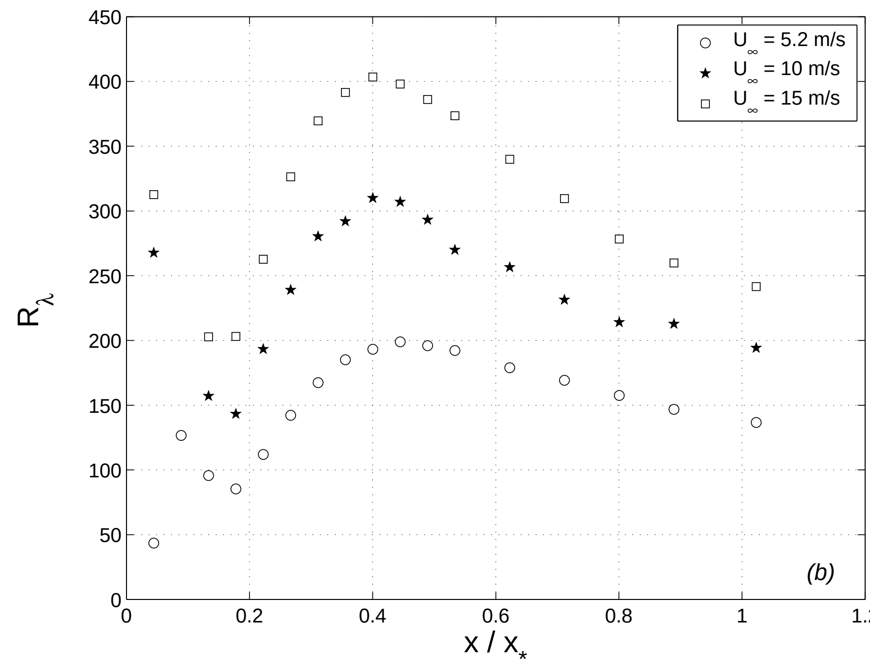
From non-gaussianity to gaussianity



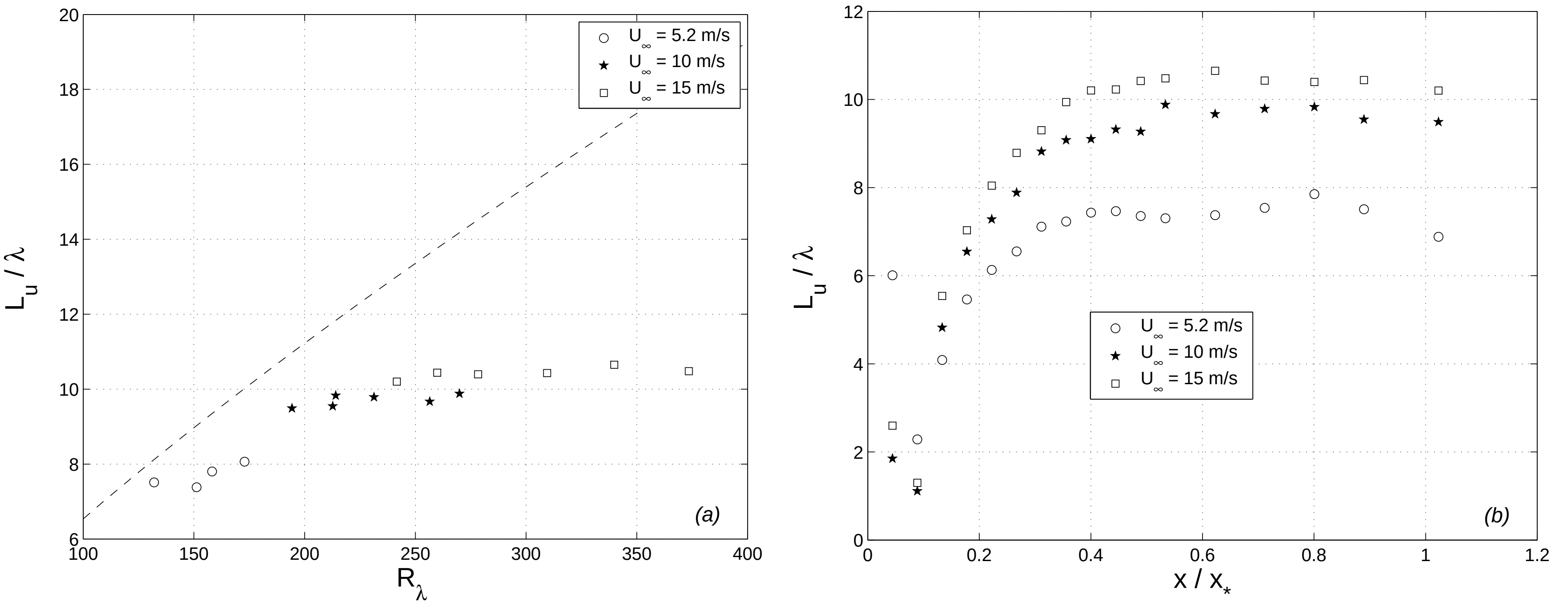
Note that S_u and F_u are close to 0 and 3 respectively at $x > x_{peak}$.

L_u/λ and Re_λ





L_u/λ versus Re_λ and Re_0



L_u/λ indep of Re_λ but $L_u/\lambda \propto Re_0^{1/2}$?

Self-preserving single-scale spectra

In homogeneous region $x > x_{peak}$,
$\frac{\partial E(k,t)}{\partial t} = T(k,t) - 2\nu k^2 E(k,t)$.

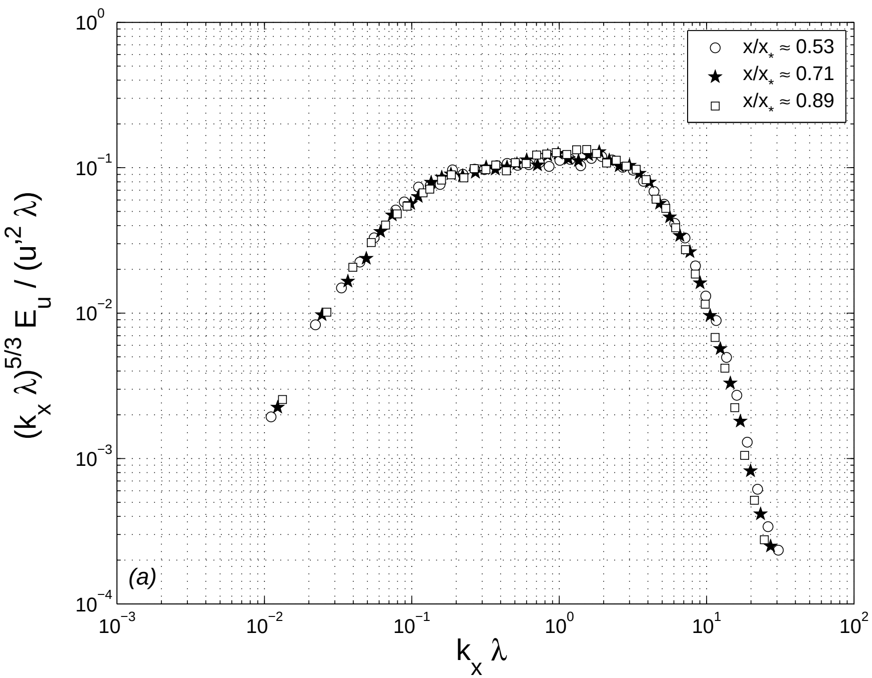
Admits exact solutions of the form
$E(k,t) = u'^2(t)l(t)f(kl(t), Re_0, *)$ and
$T(k,t) = \frac{d}{dt}(u'^2(t)l(t))g(kl(t), Re_0, *)$. (See George PoF **4**, 2192, 1992; George & Wang PoF **21**, 025108, 2009.)

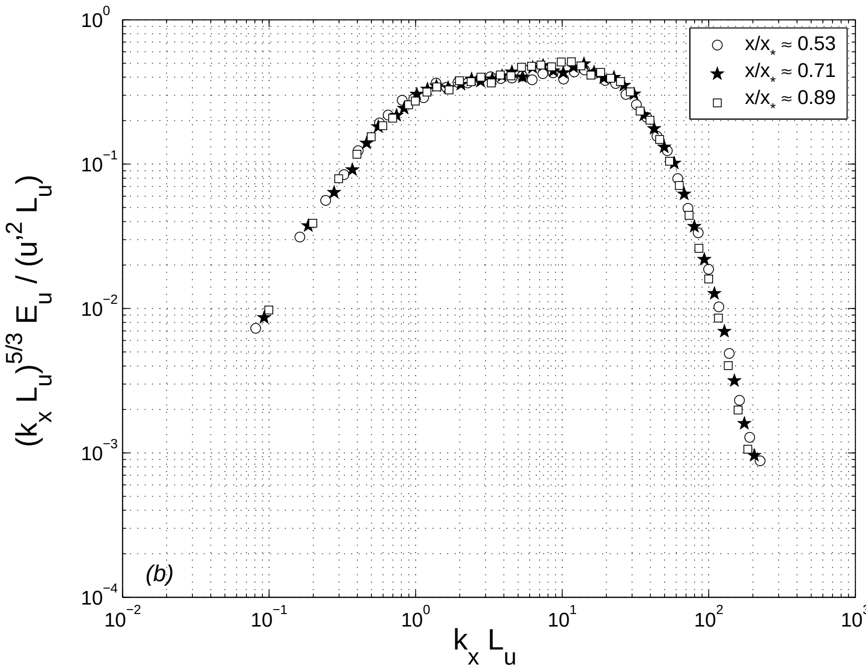
These solutions are all such that $L = \alpha(Re_0, *)l(t)$ and $\lambda = \beta(Re_0, *)l(t)$, hence L/λ remains constant during decay but can nevertheless depend on Re_0.

One particular such solution is such that l is itself constant during decay and u'^2 decays exponentially, close to what is observed. Such solutions are such that the ratio of outer (L) to inner (λ) length-scales is constant during decay even though Re_λ decays fast. As observed!

Self-preserving energy spectrum

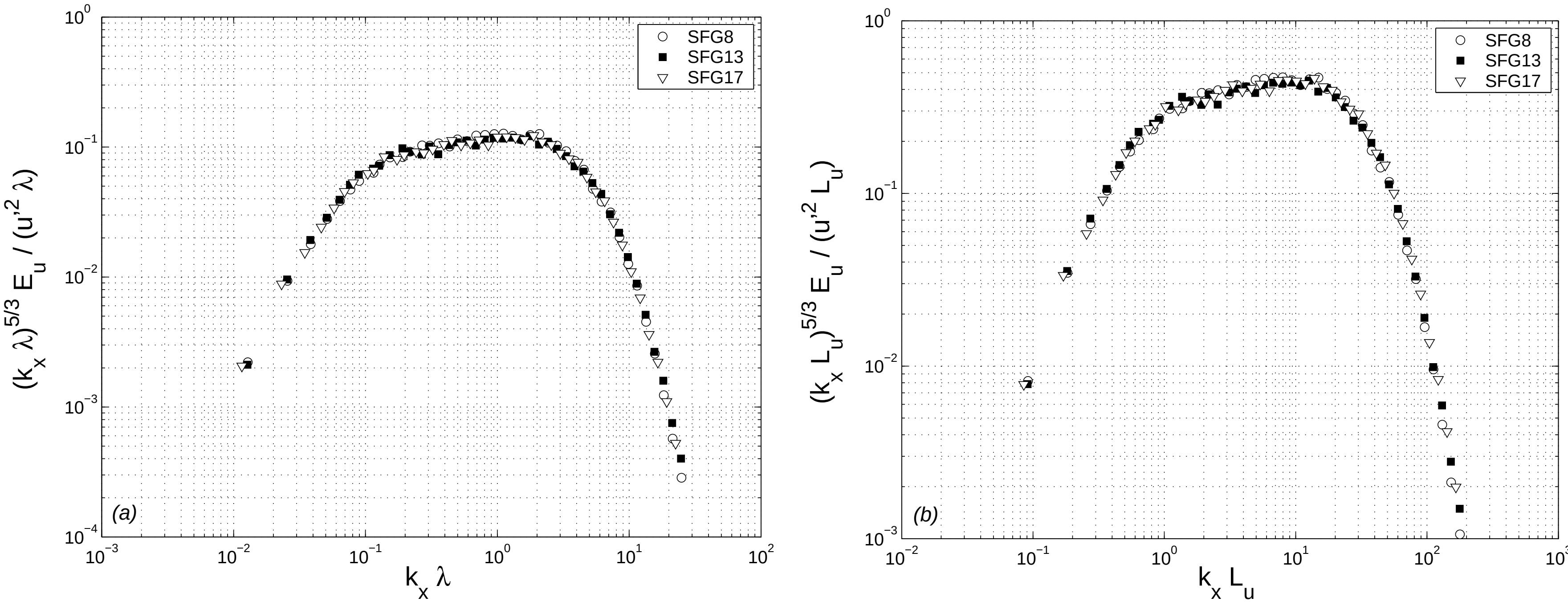
$$E_u(k_x, x) = u'^2(x) L_u f(k_x L_u, Re_0)$$





One fractal square grid and three different x positions

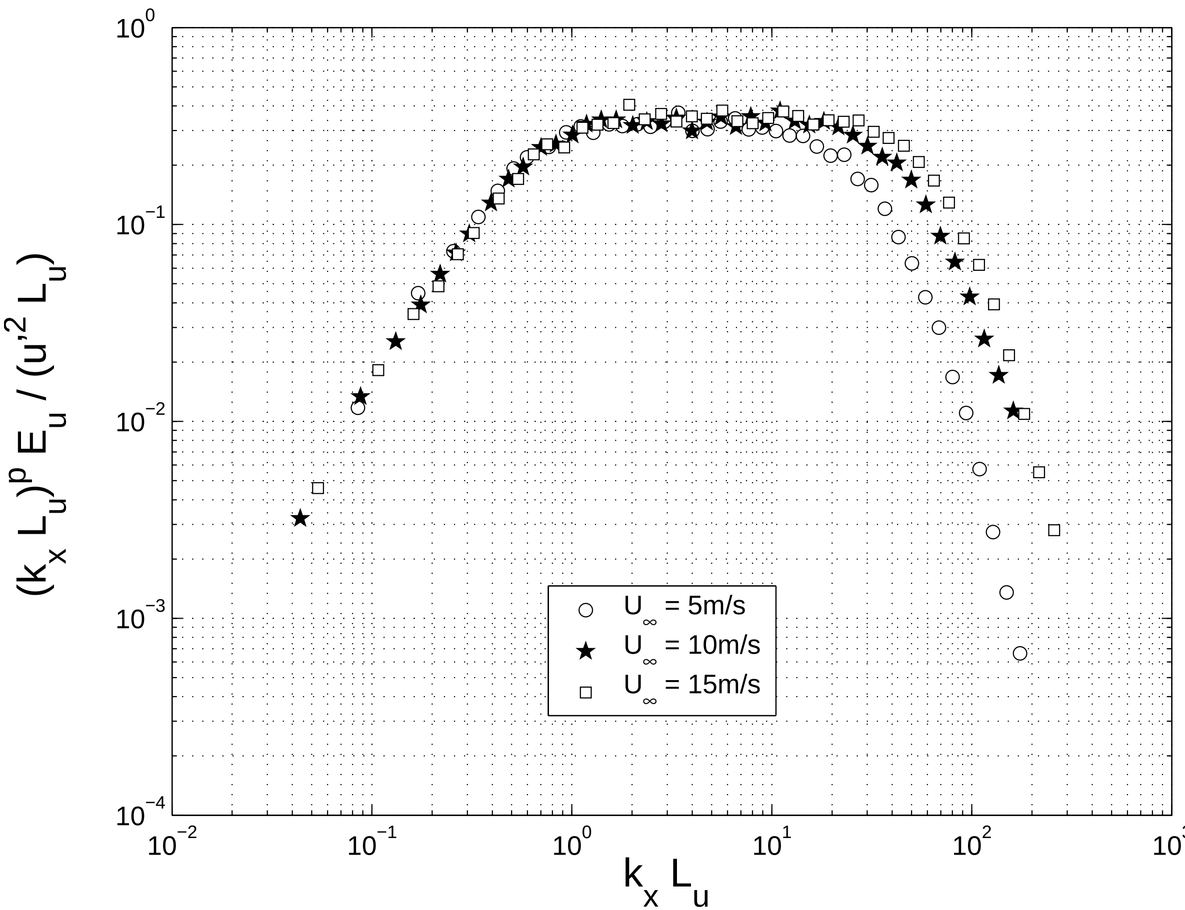
Self-preserving energy spectrum



One x/x_* position and three different fractal square grids

Energy spectrum's Re_0 dependence

$E_u(k_x, x) = u'^2(x) L_u (k_x L_u)^{-p}$ for $1 < k_x L_u < Re_0^{3/4}$

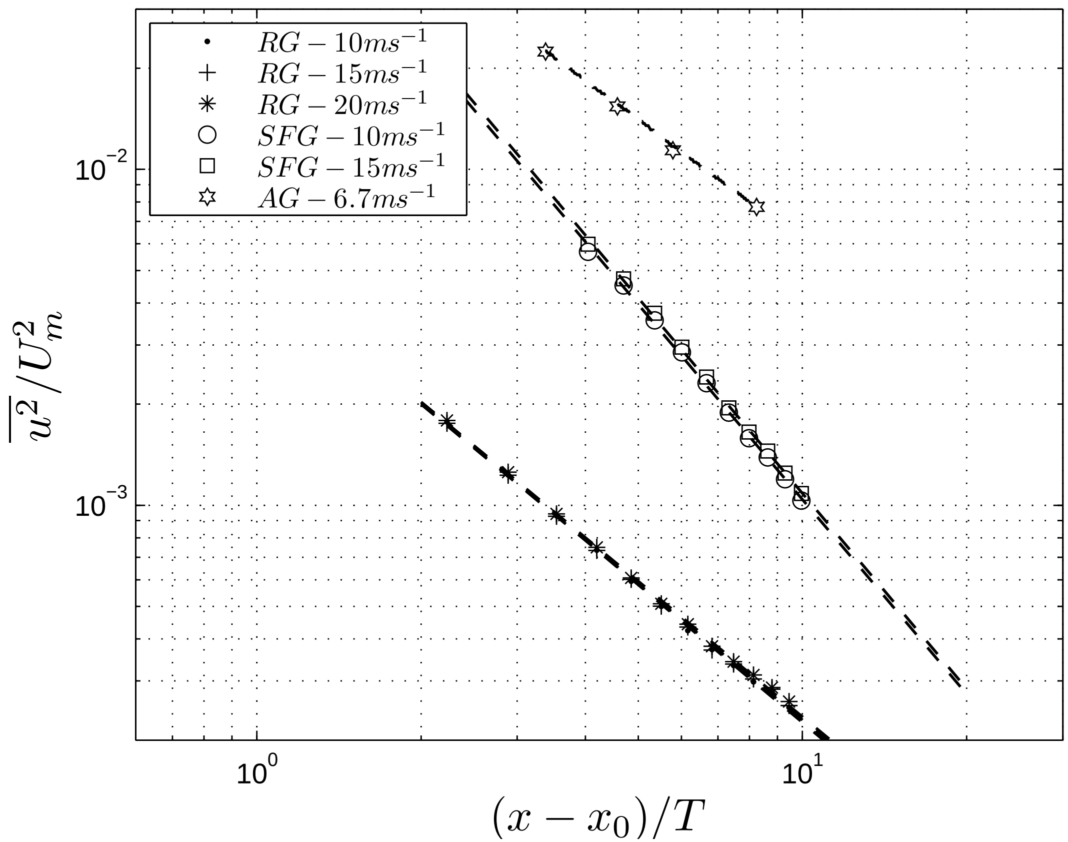


One x/x_* position, one fractal square grid, three different Re_0

p increases with Re_0, perhaps towards $5/3$

Longer streamwise fetch

From $0.5 < x/x_* < 1.0$ to $0.5 < x/x_* < 1.5$



Decay exponents: from 1.21 to 1.72 for RG and AG (Mydlarski & Warhaft '96); from 2.36 to 2.57 for FSG by various ways to fit

No universality of model constants

Consider for the fun of it the k-epsilon model equation for decaying homogeneous isotropic turbulence:

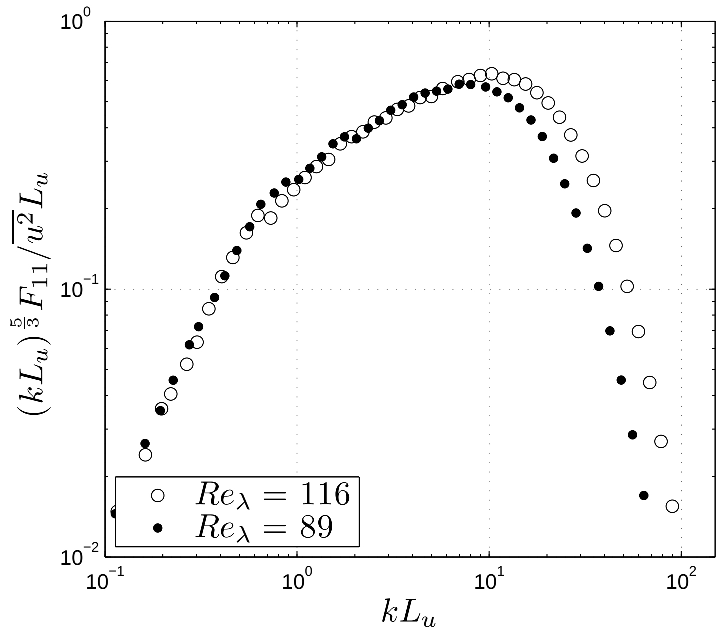
$$\frac{d}{dt}\epsilon = -\frac{2}{3}C_{\epsilon_2}\epsilon^2/u'^2$$

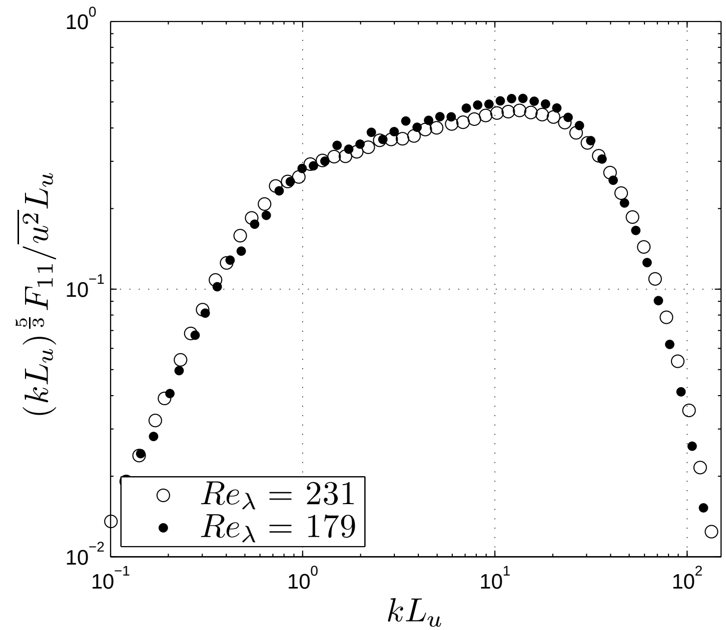
where the decay exponent is equal to $1/(C_{\epsilon_2} - 1)$.

Hence, $C_{\epsilon_2} \approx 1.8$ (close to standard value) for RG and AG, but $C_{\epsilon_2} \approx 1.4$ for FSG.

However, the more important point is the suggestion that there may be at least two different classes of decaying homogeneous turbulence: a two-scale cascading type of decaying turbulence and a single-scale self-preserving type of decaying turbulence. This is a more serious hit on universality....but the possibility exists at this stage of considering “universality classes”.....

Two classes of small-scale turbulence?





Two classes of small-scale turbulence?

A self-preserving/single-scale class where (assuming $5/3$)

(i) $E_u(k_x) \sim (\frac{u'^3}{L_u})^{2/3} k_x^{-5/3}$ for $1 \ll k_x L_u \ll Re_0^{3/4}$

(ii) $L_u/\lambda \propto Re_0^{1/2}$ but independent of x in the decay region where Re_λ decays fast. Decoupling between L_u/λ and Re_λ.

(iii) Fast turbulence decay, decay exponents around 2.5

And the K41 class where (assuming asymptotic $5/3$)

(i) $E_u(k_x) \sim (\frac{u'^3}{L_u})^{2/3} k_x^{-5/3} \sim \epsilon^{2/3} k_x^{-5/3}$ for $1 \ll k_x L_u \ll Re_\lambda^{3/2}$.

(iii) $L_u/\lambda \sim Re_\lambda$ locally at every x in the decay region.

(iv) Slow turbulence decay, decay exponents around 1.3.

What does this mean for LES modeling?

And two final thoughts...

1. Possibilities to passively design/manage bespoke small-scale turbulence for various applications?
2. What if turbulence in various cases in nature and engineering appears as a mixture of such different classes? How do we model it then?

This talk in papers

JCV, Phys. Lett A **375**, 1010 (2011)
N. Mazellier & JCV, PoF **20**, 075101 (2010)
P. Valente & JCV, preprint submitted December 2010