Turbulence in zero viscosity limit with Boundary effects

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Workshop Basic issues of extreme events in turbulence

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Two Basic Idea:

Weak convergence and Boundary Effects for Energy Dissipation

There is a strong analogy between statistical theory and weak convergence. Statistical theory mean average

1 Average over an ensemble of solutions

2 Average over different values in space or time of the same solution

3 Combine the two above points of view

And there is a belief that these different averages carry the same properties (An ergodic theorem , the Taylor hypothesis.)

Weak convergence

Weak convergence involves a family say u_ν of functions which may not converge in the usual sense but with moments again any convenient smooth test fonction $x \mapsto \phi(x)$ converging:

$$\lim_{\epsilon\to 0}\int u_\nu(x)\phi(x)dx \to \int \overline{u_\nu(x)}\phi(x)dx$$

Exemple

$$u_\nu(x) = \sin\frac{x}{\nu}\,,$$
$$\int(\sin\frac{x}{\nu})\phi(x)dx = \nu\int(\cos\frac{x}{\nu})\frac{d\phi(x)}{dx}dx \to 0\,,$$
$$\text{but}\int(\sin\frac{x}{\nu})^2\phi(x)dx \to \frac{1}{2}\int\frac{d\phi(x)}{dx}dx\,,$$
$$\text{and}\quad 0 = |\overline{u_\nu(x)}|^2 < (\overline{u_\nu(x)})^2 = \frac{1}{2}\,.$$

What I want to do

- Will not touch the “Clay problem" and assume the existence of a smooth solution $u(x,t)\, x \in \Omega, t \in [0,T]$ of the Euler with zero normal component on the boundary and consider smooth solutions of Navier-Stokes with the same initial data. *Energy estimates are much easier for finite time dependent problems.*

- The time $T > 0$ is fixed will not touch the issue of simultaneous (distinguished) limits $\nu \to 0$ and $T \to \infty$, where there are very few mathematical results. For instance “Enhanced dissipation and inviscid damping in the inviscid limit of the Navier-Stokes equations near the 2D Couette flow" Bedrossian, Masmoudi, Vicol arXiv:1408.4754

- I will use the energy estimate because is the only estimate available . Therefore only weak convergence is involved and weak convergence is the deterministic counter part of the statistical approach of turbulence. Both are based on averaging.

- Revisit a basic criteria of Kato To the best of my knowledge this is the only deterministic scenario where one can relate anomalous dissipation of energy with appearance of turbulence.

- Underline the consistency of this Kato criteria with several ansatz used in laminar or turbulent region of a fluid near an obstacle.

The Solenoidale Navier Stokes and Euler Equations in $\Omega \subset \mathbb{R}^d, d = 2, d = 3.$

The Navier -Stokes equations

$$
\begin{aligned}
&\partial_t u_\nu + (u_\nu \cdot \nabla) u_\nu - \nu \Delta u_\nu + \nabla p_\nu = 0\,,\\
&\text{In } \Omega \times [0,T] \quad \nabla \cdot u_\nu = 0\,,\\
&\text{On } \quad \partial\Omega \times (0,T) \quad u_\nu \cdot \vec{n} = 0\,, \text{ and } \quad (u_\nu)_\tau = 0\\
&u_\nu(x,0) = u(x,0) \quad Re = \frac{UL}{\nu^{\text{fluide}}} \quad \nu = Re^{-1}\,.
\end{aligned}
$$

The Euler equations

$$
\begin{aligned}
&\partial_t u + (u \cdot \nabla) u + \nabla p = 0\,,\\
&\text{In } \Omega \times [0,T] \quad \nabla \cdot u_\nu = 0\,,\\
&\text{On } \quad \partial\Omega \times (0,T) \quad u_\nu \cdot \vec{n} = 0\,,\\
&u_\nu(x,0) = u(x,0)\,.
\end{aligned}
$$

Comments

The name “incompressible" does not seems appropriate. It is not used by Leray in his founding paper. The reason is that these equations are perfectly adapted to the description of the fluctuations of density and temperature for a compressible fluide if the ratio between the fluctuation of velocity and the sound speed is small. Small Mach number. But then the only scaling parameter is the Reynolds number. In most practical applications the Reynolds number is very large (for instance, the Reynolds number is about 100 *for the air around a moving bicycle, about* 10^8 *for a moving car, and is of the order of* 10^{12} *in climate and meteorological applications). This naturally suggests to investigate and compare the behavior of the Navier-Stokes equations , for large values of the Reynolds numbers, to that of Euler equations, which are obtained formally by substituting* $Re = \infty$

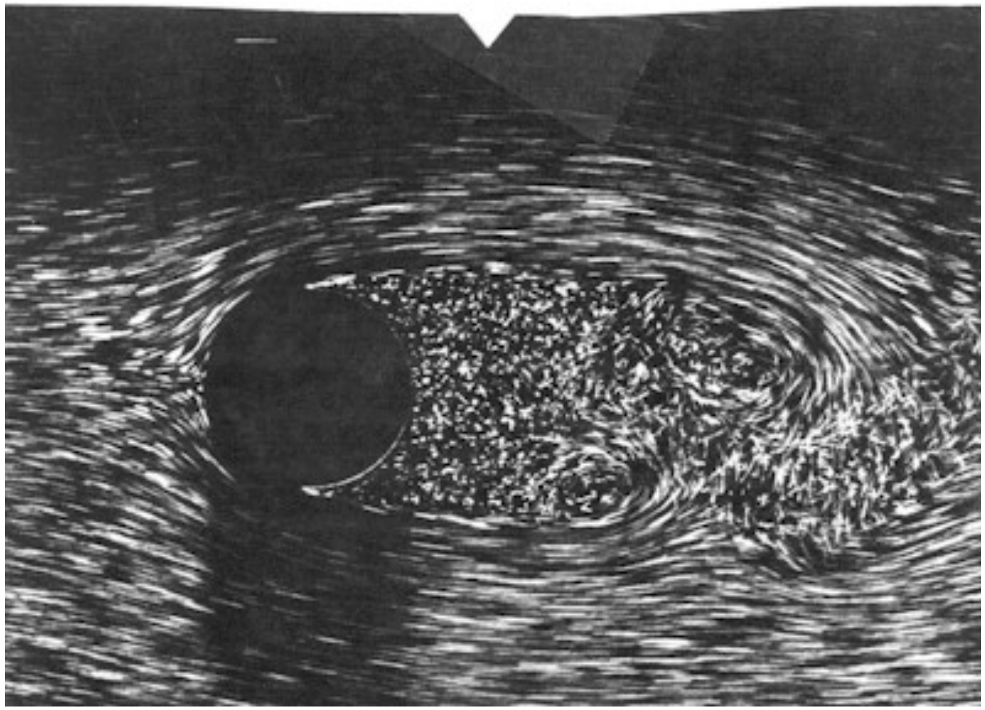
The relation between Navier-Stokes and Euler equations is trivial in the presence of a smooth solution $u(x,t)$ of Euler and in the absence of boundary effect. In the presence of boundary effects things are much more complicated.

The obvious difficulty comes from the fact that only the impermeability condition remain for $\nu \to 0$. The relation $(u_\nu)_\tau = 0$ does not persist.

Therefore the solution has to become singular near the boundary.

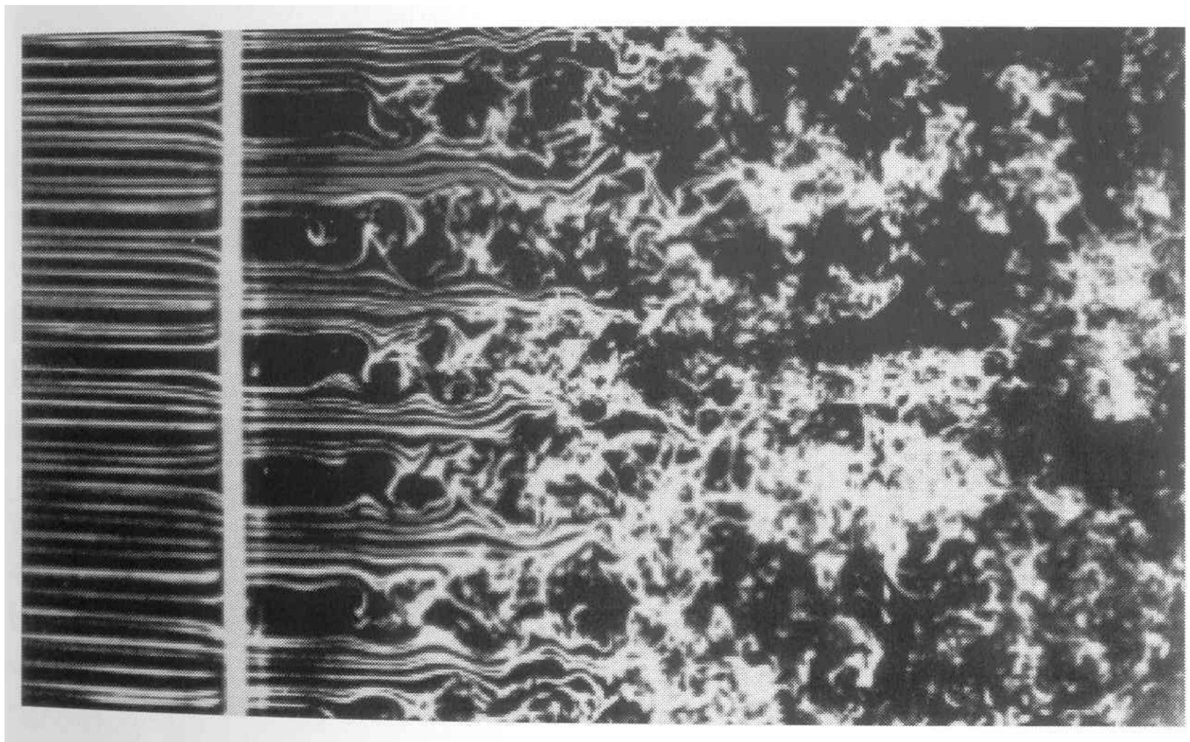
Moreover due to the non linearity of the advection term $u \cdot \nabla u$ and the effect of the pressure such singularities may propagate inside the domain as it is observed in the wake of an obstacle.

And this turn out to be the most natural effet to generate turbulence (even for homogenous turbulence far from the boundary).



Turbulent flow behind a sphere

Wake of an airplane



Grid Generated Homogeneous Turbulence

Energy balance with $u_\nu(x,0)=u(x,0)$
The ν uniform estimate.

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\nu(x,\tau)|^2 dx d\tau + \nu \int_\Omega |\nabla u_\nu|^2 dx = 0\,.$$
$$\int_\Omega \frac{|u_\nu(x,t)|^2}{2} dx + \nu \int_0^t \int_\Omega |\nabla u_\nu(x,s)|^2 dx ds = \int_\Omega \frac{|u(x,0)|^2}{2} dx\,.$$

$\overline{u_\nu}$ denotes the weak limit of a subsequence of solutions u_ν of Navier-Stokes equations.

Relative estimate:

$$\partial_t(u_\nu - u) + u_\nu \cdot \nabla u_\nu - u \cdot \nabla u - \nu \Delta u_\nu + \nabla p_\nu - \nabla p = 0$$

$$(u_\nu \cdot \nabla u_\nu - u \cdot \nabla u, u_\nu - u) = (u_\nu - u, S(u)(u_\nu - u))\,;$$

$$S(u) = \frac{\nabla u + \nabla^t u}{2}\,;$$

$$\frac{d}{dt}\frac{1}{2}|u_\nu - u|^2_{L^2(\Omega)} + \nu \int_\Omega |\nabla u_\nu|^2 dx \le |(u_\nu - u, S(u)(u_\nu - u))|$$

$$-\nu \int_\Omega (\nabla u_\nu \cdot \nabla u) dx + \nu \int_{\partial\Omega} \partial_{\vec{n}} u_\nu u d\sigma\,.\ \text{The bad term!}\,.$$

Without boundary u_ν converges to u in $C(0,T;L^2(\Omega))$. Otherwise the situation is much more subtle!!!

About weak convergence:

For any $\overline{u_\nu}$ weak $L^\infty((0,T) : L^2(\Omega))$ limit of a sequence of solutions of Navier-Stokes equations one has has following standard Hilbert type-properties:

$$\overline{u_\nu} \in C_{\mathsf{weak}}(0,T;L^2(\Omega))\,, \quad \overline{u}_\nu(x,0) = u_0(x)\,,$$
$$\text{in } \Omega \times (0,T) \quad \nabla\cdot\overline{u_\nu} = 0\,; \qquad \text{on } \partial\Omega\times(0,T) \quad \overline{u_\nu}\cdot\vec{n} = 0\,,$$

$$\forall t>0 \quad \int_\Omega |\overline{u_\nu}(x,t)|^2dx \le \overline{\int_\Omega |u_\nu(x,t)|^2dx} \le \int_\Omega |u_0(x)|^2dx\,,$$

$$0 \le \overline{\nu\int_0^T\int_\Omega |\nabla_x u_\nu(x,t)|^2dxdt} \le 2(\int_\Omega |u_0(x)|^2dx - \overline{\int_\Omega |u_\nu(x,T)|^2dx})\,.$$

The 1983 Kato Theorem.

Theorem the following facts are equivalent:

$$u_\nu(t) \to u(t) \text{ in } L^2(\Omega) \text{ uniformly in } t \in [0,T]\,, \tag{1}$$

$$u_\nu(t) \to u(t) \text{ weakly in } L^2(\Omega) \text{ for each } t \in [0,T]\,, \tag{2}$$

$$\lim_{\nu\to 0} \nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dx dt = 0\,, \tag{3}$$

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}} |\nabla u_\nu(x,t)|^2 dx dt = 0\,. \tag{4}$$

and eventually the fact that for all tangent to the boundary vector field $w(x,t) \in \mathcal{D}((0,T)\times\partial\Omega)$ one has:

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} \frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t) w(\sigma,t) d\sigma dt = 0\,. \tag{5}$$

The proof: An updated version of the basic result of Kato

(1)$\Rightarrow$ (2) and (3) $\Rightarrow$ (4) trivial.

With the energy conservation for u and

$$\int_\Omega |\overline{u_\nu}(x,t)|^2 dx \leq \overline{\int_\Omega |u_\nu(x,t)|^2 dx} \leq \int_\Omega |u_0(x)|^2 dx$$

(3) is a consequence of (2).

To deduce that (4) implies (5) , (in the Dirichlet case for sake of simplicity) construct a family of divergence free vector fields $\widehat{w}_\nu \in \mathcal{C}^\infty(\overline{\Omega} \times (0,T))$ *with support in* $\{(x,t) \in (d(x,\partial\Omega) < \nu \ \times (0,T))\}$ *which coincides with* w *on the boundary and with gradient bounded in* L^∞ *by* $C\nu^{-1}$*.*

Boundary coordinates $\{(\sigma,t) \in \partial\Omega \times (0,T)\,, s = d(x,\partial\Omega)\}$,

$w \in L^{lip}(\partial\Omega \times (0,T))$; $\quad w \cdot \vec{n} = 0$;

$\Theta \in \mathcal{D}(\overline{0,1})$, $\quad \Theta(0) = 0\,, \quad \Theta'(0) = 1$;

$$\widehat{w}(x,t) = \nabla \wedge ((\vec{n}(\sigma) \wedge w(\sigma,t)\nu\Theta(\frac{s}{\nu}))$$

$\Rightarrow$ on $\quad \partial\Omega \quad \widehat{w}(x,t) = w(x,t)$

$\Rightarrow$ In $\quad \Omega\,, \qquad \nabla \cdot \widehat{w} = 0\,,$ $\quad$ and $\quad$ support $\quad \widehat{w} \subset \{d(x,\partial\Omega) < \nu\}$;

$\Rightarrow \|\nabla_x \widehat{w}\|_\infty \le C\nu^{-1}\,, \sup_t \|\nabla_x \widehat{w}(x,t)\|_{L^2(\Omega)} \le C\nu^{-\frac{1}{2}}\,.$

Multiplication of the Navier-Stokes equation and integration by part provides the formula

$$\nu \int_{\partial\Omega} \frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t) w(\sigma,t) d\sigma =$$
$$\nu(\nabla u_\nu, \nabla \widehat{w_\nu})_{L^2(\Omega)} - (u_\nu \otimes u_\nu, \nabla \widehat{w_\nu})_{L^2(\Omega)} + (\partial_t u_\nu, \widehat{w_\nu})_{L^2(\Omega)} .$$

Then

$$(1) \lim_{\nu \to 0} \nu \int_0^T \int_{\{x \backslash d(x,\partial\Omega)<\nu\}} |\nabla u_\nu(x,t)|^2 dx dt = 0$$

(2) Poincaré estimate

$$(3) |\nabla_x \widehat{w}|_\infty \le C\nu^{-1}, \sup_t \|\nabla_x \widehat{w}(x,t)\|_{L^2(\Omega} \le C\nu^{-\frac{1}{2}}$$

$$\Rightarrow \lim_{\nu \to 0} \nu \int_0^T \int_{\partial\Omega} \frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t) w(\sigma,t) d\sigma dt = 0$$

$$\Rightarrow \lim_{\nu \to 0} \nu \int_0^T \int_{\partial\Omega} \frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t) u(\sigma,t) d\sigma dt = 0 .$$

with

$$\frac{1}{2}\frac{d}{dt}|u_\nu - u|^2_{L^2(\Omega)} + \nu \int_{\partial\Omega} |\nabla u_\nu|^2 dx$$

$$\leq |(u_\nu - u, S(u)(u_\nu - u))| + \nu \int_{\partial\Omega} |\nabla u_\nu||\nabla u| dx$$

$$+\nu \int_{\partial\Omega} \partial_{\vec{n}} u_\nu u d\sigma$$

(5) implies (1) and this concludes the proof.

However cases where the Kato criteria does not apply seems to be the general situation rather than the exception.

It corresponds to real or numerical observations. It is the most common way of generating turbulence.

Kato Criteria and d’Alembert paradoxe.

Total force applied by the air on a wing (lift $+$ drag) W (with $\Omega = \mathbb{R}^d \backslash W$)

$$F = -\int_{\partial\Omega} p\vec{n}d\sigma \,.$$

A simple computation done in the permanent regime, for $W \subset \mathbb{R}^3$ bounded shows that if p is given by the Euler equation one has $F = 0$. This implies that if motion of the air is given by the Euler equation the plane cannot fly! The classical “d’Alembert paradoxe." Resolving such paradox was one of the main motivation for the introduction of the Navier-Stokes equations.

If u_ν would strongly converge to u same would be true for p_ν and the conclusion would be that for ν small enough (which corresponds to realistic Reynolds numbers) the force exerted on the wing would be arbitrarily small. Not so good for flying!

Turbulence in the absence of Kato Criteria.

In turbulence theory the word anomalous dissipation of energy refers to the following anomaly.

$$\liminf_{\nu\to 0} \nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dx dt = \epsilon > 0 \tag{6}$$

always associated to turbulence. Kato criteria is the only statement (in classical functional analysis) connecting turbulence and anomalous dissipation of energy.

The non convergence to a smooth solution is equivalent to the anomalous dissipation of energy near at least one neighborhood U of a point $(x_{\mathrm{tur}}, t_{\mathrm{tur}}) \in \partial\Omega \times [0,T]$ such that:

$$\liminf_{\nu\to 0} \nu \int_0^T \int_{x,\{d(x,\partial\Omega)<C\nu\}\cap U} |\nabla u_\nu(x,t)|^2 dx dt = \epsilon > 0 \tag{7}$$

The sequence u_ν will be turbulent in such region!

Other Local Definitions of Turbulence

If one of the equivalent form of the Kato criteria is not satisfied and in particular if there exists anomalous energy dissipation the following statements are also equivalent:

1 $\overline{u_\nu} \neq u$ in $\Omega \times (0,T)$

2 For any $n > 0$ there is an open set $U^n \subset \mathbb{R}^d \times (0,T)$ such that $U^n \cap \partial\Omega \times (0,T) \neq \emptyset$ and $\overline{u_\nu} \notin C^{0,\frac{1}{n}}(U^n \cap \Omega \times (0,T))$.

3 There is at least one point $p_{\text{turb}} = (x_{\text{turb}}, t_{\text{turb}}) \in \partial\Omega \times [0,T]$ such that for any neighborhood U of p_{turb} and any n one has:

$$\sup_{\nu \to 0} \|u_\nu\|_{C^{0,\frac{1}{n}}(U)} = \infty$$

Local description of high Reynold number flow past an obstacle and Kato Criteria

The above introduction of the turbulent part of the boundary makes a natural connection with the approach of engineers which for high number Reynolds flow past an obstacle decompose the boundary in several pertinent part involving different scalings and related ansatz.

This construction belongs much more to the art of the engineers than to the reasoning of a mathematician. Moreover since it involves unstable phenomena it leads lead to ill posed or at least very unstable problems.

Experiment, numerical simulation and phenomenological argument concern mostly time independent problem. However their counter part for slowly varying time dependent flow should carry the same structure.

Therefore below I will omit the time variable, but AS A CONCLUSION emphasize the similitudes between the scalings involved and the Kato Criteria.

The 3 basic regions

- The Prandlt laminar boundary layer.

- The recirculation laminar boundary layer and the triple deck ansatz.

- The Von-Karman Prandlt turbulent layer.

The Prandlt 1904 Boundary Layer

The first idea Prandlt (1904) (which has been later extended to a huge class of problems which turn ou tot be less challenging than the original one..they are often linear) was to represent the solution near the boundary by parabolic boundary layer

$$u_\nu \simeq U_\tau(\frac{d(x,\partial\Omega)}{\sqrt{\nu}},x_\tau) + \sqrt{\nu}U_{\vec{n}}(\frac{d(x,\partial\Omega)}{\sqrt{\nu}},x_\tau) \tag{8}$$

This leads to the Prandlt equation.

This system is not always well posed (even for arbitrarily small time). A serie of sufficient conditions for the well posedness (in particular for the time dependent problem for short time) have been proposed: Some simple configuration like the shear flow or the rotating flow as above, some analyticity hypothesis, some monotonicity hypothesis or a mixture of both cf. Kukavica, Masmoudi, Vicol and Wong Arix.org 2014. for recent results and updated references.

The Triple Deck Ansatz

In the permanent regime the first break down of the Prandlt representation appears when $x_\tau \mapsto u_\tau(x_\tau, x_{\vec{n}})$ cease to be monotonic and in particular at at a separation point $x^* \in \partial\Omega$ characterized by:

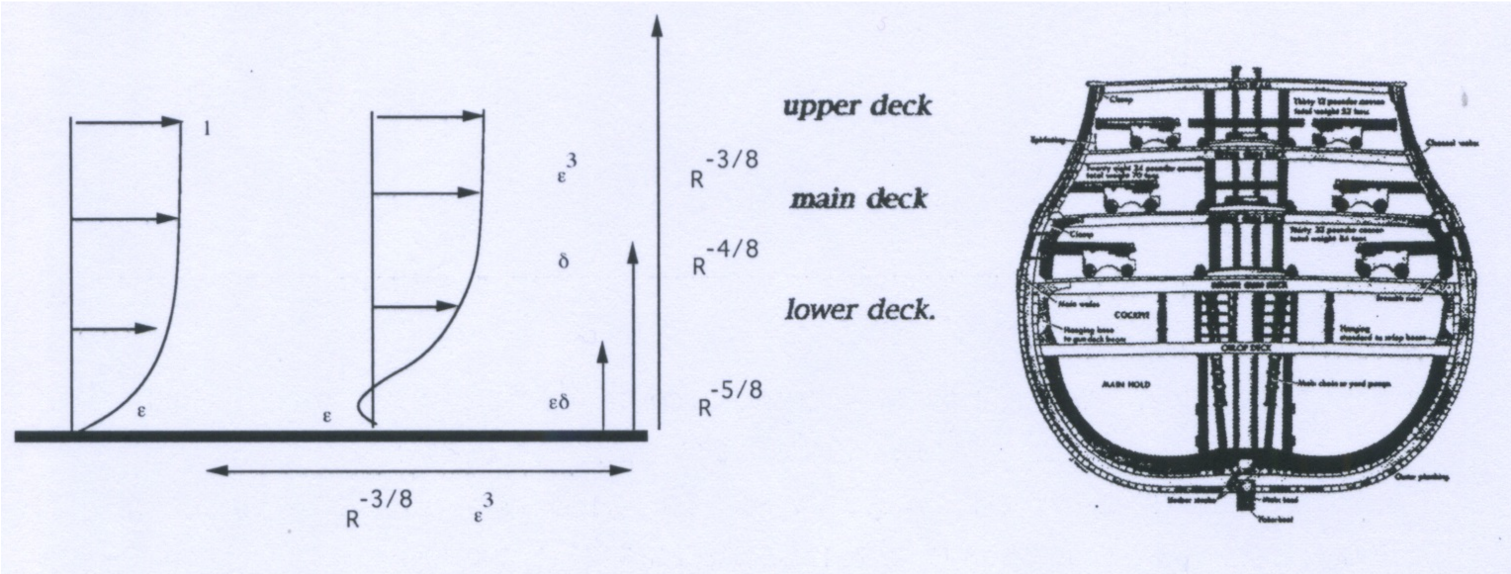
$$\partial_{\vec{n}} u_\tau(x - x^*) \simeq \sqrt{|x - x^*|}$$

cf Landau and for a complete proof in the stationary regime near a half line Dalibar and Masmoudi (work in progress).

However the fluid may remain laminar after this point in some neighborhood $\mathcal{V}$ of a part of the boundary.

Hence to describe this structure in some neighborhood $\mathcal{V}$ of the boundary a refined analysis of the Prandlt boundary layer $\{x, d(x, \partial\Omega) < \sqrt{\nu}\}$ has been proposed by Stewartson K. (1974). It carries the name of triple deck.

The Triple Deck Interpretation with the Porthmouth



1 In the Upper Deck $\{x\backslash \quad \sqrt{\nu} < d(x,\partial\Omega)\} \cap \mathcal{V}$ the solution is described by the Euler flow.

2 In the Lower Deck $\{x\backslash \quad 0 < d(x,\partial\Omega) < \nu^{\frac{5}{8}}\} \cap \mathcal{V}\}$ the solution is described by the above Prandlt boundary laye ansatz.

3 In Middle Deck $\{\nu^{\frac{5}{8}} < d(x,\partial\Omega) < \sqrt{\nu}\} \cap \mathcal{V}$ which connects the two above regions the following scaling is proposed.

$$u_\nu(x) \simeq (\nu^{\frac{1}{8}} U_\tau(\frac{d(x,\partial\Omega)}{\nu^{\frac{5}{8}}}, \frac{x_\tau}{\nu^{\frac{3}{8}}}), \nu^{\frac{3}{8}} U_{\vec{n}}(\frac{d(x,\partial\Omega)}{\nu^{\frac{5}{8}}}, \frac{x_\tau}{\nu^{\frac{3}{8}}})) \tag{9}$$

Prandlt Boundary layer , the Triple Deck and Kato criteria

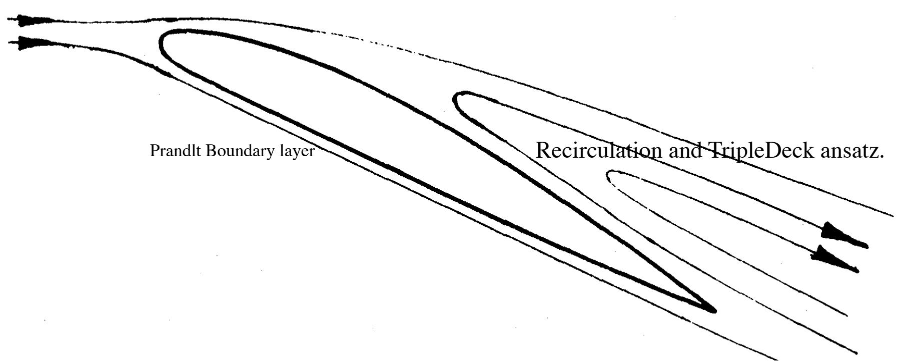
Observe that both the Prandlt boundary layer ansatz and the triple deck ansatz describe the fluide in a region at the distance $\sqrt{\nu}$ of the boundary. Moreover if these two ansatz would give an accurate description of fluid u_ν all around the boundary then one would have (explicit computation!)

$$\nu \int_{\{x, \backslash d(x,\partial\Omega)<\nu^{\frac{1}{2}}\}} |\nabla u_\nu(x,,t)|^2 dxdt \leq C\nu^{\frac{1}{2}} \to 0 \tag{10}$$

In agreement with the Kato criteria and with the figure below that would describe convergence toward a laminar regime with recirculation.

Laminar regime with recirculation

Laminar regime



The Prandlt -Von Karman 1932 turbulent layer

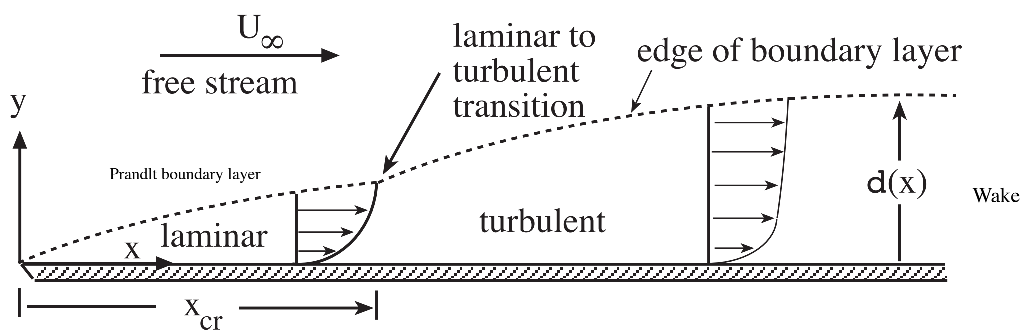
Since convergence to a smooth solution is not expected , a turbulent boundary layer for $\overline{u_\nu}$ should be present in general around some part of the boundary.

The only thing available is a description based on experiment, numerical analysis and dimension analysis. It is the Von Karman-Prandlt turbulent layer (1932) . It provides an ansatz for the tangential component of the velocity $u_\tau(x_{\vec{n}}, x_\tau)$ in the layer

$$B_{\text{turbulent}} = \{x, d(x, \Omega) < \nu\} \cap \mathcal{W}$$

with $\mathcal{W}$ denoting a neighborhood of a part of the boundary .

Appearance of turbulent layer and wake



The Prandlt -Von Karman 1932 turbulent layer and Kato criteria

On $\partial\Omega \cap \overline{\mathcal{W}}$ the quantity

$$u^* = \sqrt{\nu \partial_{\vec{n}} u_\tau} \tag{11}$$

which has the dimension of a velocity, is assumed to be of the order of unity.

Then in $B_{\text{turbulent}}$ one has:

$$u_\tau(x_{\vec{n}}, x_\tau) = u^* U_\tau(s)\,, \quad s = u^* \frac{x_{\vec{n}}}{\nu} \tag{12}$$

with $U_\tau(s)$ an intrinsic function of the “number " s . With phenomenological argument this function is almost linear for $0 < s < 20$ and given by a Prandlt -Von Karman wall law

$$U_\tau(s) = \kappa \log s + \beta \quad \text{for } 20 < s < 100\,. \tag{13}$$

However either with (11) which implies that

$$\nu\partial_{\vec{n}}(u_\tau)_{|\partial\Omega} \geq \alpha > 0$$

or with (13) which implies

$$\nu\int_{\{x,\backslash d(x,\partial\Omega)<\nu^{\frac{1}{2}}\}} |\nabla u_\nu(x,,t)|^2 dx \geq \epsilon > 0 \qquad (14)$$

one observes that the existence of such boundary layer is consistent with the fact that $\overline{u_\nu}$ does not converge to u or is not in $C^{0,\alpha}$ (for any α) in some neighborhood of a part of the boundary.

And of course this is necessary for the appearance of a turbulent wake.