FROM MICROSCOPIC TRAFFIC MODELS TO FLUID TRAFFIC FLOW MODELS

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OUTLINE

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. 1A: INTRODUCTION

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. 2: LAGRANGIAN MACROSCOPIC VIEW

. 3: EULERIAN MACROSCOPIC VIEW

. 4: CASE WITH RELAXATION TERMS: ...

. 5: COMMENTS ... MORE DETAILS ... NOT IN THESE NOTES...
MOTIVATION

Introduce a general class of second-order models (AR)...

in order to 
. be correct at small scales ($\sim 1$ m, 1 sec):
   - no crush, no negative velocities...
   - be rigorously based on microscopic models: SCALES
   - be very general and flexible:
     \[ \partial_t \rho + \partial_x (\rho u) = 0 \]
     \[ \partial_t w + \partial_x (w \nu) = 0 \]
     (initially \( w = \nu + \rho(\rho) \))
     - without external destination...
   - can be rigorously homogenized \( \rightarrow \) with P. Bagnaia
     - SIMA 2003
     - with H. Heine
     - SIMA 2006
   - preserves HJ approach as 1st order models

Prepare the discussion for round tables...

Is this AR class too stable (at larger scales)? Answer: Yes! [How much] can we violate the subcharacteristic condition? Yes, but cleverly!

Introduce a few mathematical ingredients for discussion:

* Invariant regions: Chuah, Coury-Smoller
* Subcharacteristic condition
* BV estimates for Temple systems
* CV of numerical schemes to weak (entropy) solutions
1-A: Introduction: Essentially, 3 classes of models

- Microscopic models: ODE follow the leader Models...
  - Kinetic: more considered here
  - Fluid:
    - 1st order Models: \( (LWR) \partial_t p + \partial_x (p v) = 0 \)
      - equilibrium model
      - Very robust, (too) simple
      - but associated Hamilton-Jacobi eq...
  
  - 2nd order: 1) Payne-Whitman: of gas dynamics
    - Daganzo (Requiem, 95)
    - PW is a terrible model!
    - # 1: \( \exists \) cases where \( n < 0 \)!!
    - # 2: \( v_2 = n + c(p) > n \): some information travels faster than cars!!

2) Fixing: Aw-Rascle (Resurrection, SIA 2000), Zhang (LWR)

- In RHS, replace \( \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \). Set \( \frac{\partial}{\partial t} (\rho) \leftarrow \frac{\partial}{\partial x} (\rho v) \).

\[ \begin{align*}
\partial_t \rho + \partial_x (\rho v) &= 0 \\
\partial_t (v + c(p)) + v \partial_x (v + c(p)) &= \begin{cases} 0 & \text{(Initial)} \\ \text{Relax} \rightarrow \ldots \end{cases}
\end{align*} \]

\( w \): Lagrangian marker (with) influence on \( v \)...
\[ \begin{aligned}
&\dot{x}_j(t) = \frac{v_j}{\Delta x}, \quad \text{Gazis-Herman Rotating type} \\
&v_j(t) = -P'( \frac{x_{j+1}(t) - x_j(t)}{\Delta x} ), \quad \frac{v_{j+1}(t) - v_j(t)}{\Delta x} \\
&\text{(ODE)} \quad + \frac{A}{T^2} \left( \sqrt{\frac{1}{\mu} \left( \frac{x_{j+1}(t) - x_j(t)}{\Delta x} \right)} - v_j(t) \right) \end{aligned} \]

Assume: \( A = 0 \) ≠ Bando's model, \( \mu \) subcharacteristic condition (Whitham)...

Adimensional density / specific volume...

\[ T_j = \frac{x_{j+1} - x_j}{\Delta x} \]

\( \dot{x}_j(t) = \frac{1}{T_j} \left( \frac{x_{j+1} - x_j}{\Delta x} \right) \Rightarrow \dot{\bar{x}}_j(t) = \frac{v_{j+1} - v_j}{\Delta x} \]

\[ \dot{v}_j(t) = \frac{v_{j+1} - v_j}{\Delta x}, \quad \frac{v_{j+1} + P(T_j)}{T_j} (t) = 0 \]

Remark:

- Density \( \rho = \frac{1}{\mu} \) is local, adimensional \( \Rightarrow \) unit independent! \( \rho \neq \text{unit} \)!
- Later, hyperbolic scaling \( (x, t) \Rightarrow (x', t') = (\varepsilon x, \varepsilon t), X' = \varepsilon X, \Delta X' = \varepsilon \Delta X \)
  \( \Rightarrow \) in this zoom, \( \dot{x}_j, \frac{A}{T^2}, \frac{A}{T^2} \) are scale independent
- \( X_j = j \Delta X = \sum_{k=1}^{j} \frac{x_{k+1}(t) - x_k(t)}{\Delta x} \), \( \Delta X \sim \int_{0}^{1} p(y, t) \; dy = X(x, t) \)
Remark: $\tau$ is additive, not $\rho$ !! (single lane). Eq. $5m = \Delta X$

Before

\[
\frac{A}{A} \quad \frac{A}{A} \quad \frac{A}{A}
\]

(old) $\tau_0 = \frac{30}{5} = 6$

(new) $\tau_0' = \frac{20}{5} = 4$

$\tau + \frac{1}{2} = \frac{10}{5} = 2$

$6 = 4 + 2$

$\frac{6}{2} \neq \frac{4}{2} + \frac{1}{2}$

Do it periodically, with same velocity $v$ for all cars.

Homogenized flow: average $\tau = \frac{1}{2} (4+2) = 3$. Equivalent averaged density: $\bar{\rho} = \frac{1}{\tau} = \frac{1}{3}$, $\rho = \frac{1}{\bar{\tau}}$!! 

\[\tau_A = \frac{1}{\rho_A}, \quad \tau_B = \frac{1}{\rho_B}, \quad \tau = \frac{\tau_A + \tau_B}{2}\]

see Aw, Klein, Hatine, R. SIAP, 02

Bagnerini, R. SIHA, 03

Herty, R. SIHA, 06

Houtui, R., NTH, 06
Remark:

Simplest numerical approximation: 1st order Euler explicit scheme

\[ \begin{align*}
    \dot{w}_j(t) &= \frac{w_{j+1} - w_j}{\Delta X} \\
    w_j(0) &= 0 \\
    w_0^0 &= v_0 + P(0)
\end{align*} \]

\[ \begin{align*}
    \frac{w_{j+1}^n - w_j^n}{\Delta t} &= \frac{v_{j+1} - v_j}{\Delta X} \\
    w_j^{n+1} &= w_j^n = \ldots = w_0^0 = w_0
\end{align*} \]

\[ \text{ex. } P(T) = v_{\text{max}} - V_{eq}(T) \Rightarrow P'(T) = -V_{eq}'(T) < 0. \]

Original system is with delay. Delay vanishes in our scaling,

(see below). Numerically, delay \( \Rightarrow \) explicit scheme.

In the sequel, \((\text{FD}) = \) Godunov scheme for Lagrangian continuous system \((\text{L})\)

Unile system: 1 min; 1.5 km (# 1 mile); \( v = \frac{1.5 \text{ km}}{\text{min}} = 1 \frac{\text{m}}{\text{sec}} \)

Good mesh size:

\( \Delta X = 5 \text{ m} = \frac{1}{300} \times (1.5 \text{ km}) \), \( \Delta t = 0.2 \text{ sec} = \frac{1}{300} \frac{\text{km}}{\text{hr}} \)

Quite reasonable for continuous model.
\textbf{Lagrangian macroscopic view:}

1. **Hyperbolic scaling:** \((x, X, t, \Delta X) \rightarrow (x', X', t', \Delta X') = (x, x', e_1, t, e_1 \Delta X)

2. Start from (FD), in rescaled coordinates. Then, at least formally,

\[
\frac{\tau_{n+1} - \tau_n}{\Delta t'} = \frac{\nu_{n+1} - \nu_n}{\Delta x'} \quad \text{and} \quad \frac{w_{n+1} - w_n}{\Delta t'} = 0
\]

when \(\Delta x' = e \Delta X \rightarrow 0\)

\[
\Delta t' = e \Delta t \rightarrow 0
\]

\[
\Delta t' = \frac{C}{\Delta x'}
\]

+ CFL condition

3. With \(t_n' = n \Delta t' = t, \quad x_n' = j \Delta x' = x\)

and corresponding initial data \(U_0^e(X') = U_0^e(X', X') = U_0^e(x, eX)\)

\[\Delta t' \text{ is already 0} \]

4. Start from (ODE): \[
\frac{\partial \tau}{\partial t'} = \frac{\nu_{n+1} - \nu_n}{\Delta x'} \quad \text{and} \quad \frac{\partial w}{\partial t'} = 0 \quad \text{for slow}
\]

- \(\Delta x' \rightarrow 0\)

Linear system is:

\[
\begin{cases}
\partial_t \tau - \partial_x \nu = 0 \\
\partial_t w = 0 \\
\text{Diagonalize: } \partial_t \nu + P'(t) \partial_x \nu = 0 \\
\partial_t w = 0
\end{cases}
\]

\(\nu, w: \text{Riemann invariants}\)
Eigenvalues: $A_1 = P'(t) < 0 = A_2$; $A_3$ genuinely nonlinear.

- $A_1$: shock (braking) or rarefaction waves (acceleration)
- $A_2$: linearly degenerate (LD)
- contact discontinuities
- cars follow each other, at same speed

Riemann Pb:

$L \rightarrow 1$ waves: $w = w_x$

$2$ waves:

- $w = w_x$ (contact) or
- $w = w_{\text{shock}}$

Range:

- $\text{contact}$ if $N_x < N_{x+}$
- $\text{acceleration}$ if $N_x > N_{x+}$

$A = \frac{P'(t) - P'(t^*)}{t^* - t} = \frac{N_x - N}{t^* - t}$

Remark: compare with Payne-Whitham dynamics!!!
Lagrangian Godunov scheme:

\[ \frac{\partial}{\partial t} U - \frac{\partial}{\partial x} \cdot V = 0, \quad V = w - P(T) \]

\[ \frac{\partial}{\partial t} w = 0 \]

\[ U = ( \frac{T}{w} ) \Rightarrow ( \frac{V}{w} ) \]

At \( t = t_m \), \( U( X_{i+\frac{1}{2}}, t_m ) \equiv U^m_{i+\frac{1}{2}} \) on \( ( X_{i-\frac{1}{2}}, X_{i+\frac{1}{2}} ) \)

Integrate over \( ( X_{i-\frac{1}{2}}, X_{i+\frac{1}{2}} ) \times ( t_{m-1}, t_m ) \) \[ \int_t^T \int_{X_{i-\frac{1}{2}}}^{X_{i+\frac{1}{2}}} \frac{\partial}{\partial t} U \, dx \, dt = 0 \]

\[ 0 = \Delta X \cdot \frac{\partial}{\partial \rho} + \Delta X \cdot \frac{\partial}{\partial \rho} + \Delta \rho \cdot \frac{\partial}{\partial \rho} \]

\[ \Rightarrow U^{m+1}_{i+\frac{1}{2}} = U^m_{i+\frac{1}{2}} \]

\[ U^{m+1}_{i-\frac{1}{2}} = U^m_{i-\frac{1}{2}} \]

**Finite speed of propagation:** If CFL condition satisfied, \( \Delta t \ll \frac{\Delta x}{\lambda} \)

Riemann Problem \( U( X_{i+\frac{1}{2}}, t_0 ) = \begin{cases} U^m_\rho = U^-, & X < X_{i+\frac{1}{2}} \\ U^{m+1}_\rho = U^+, & X > X_{i+\frac{1}{2}} \end{cases} \)

\[ \Rightarrow (\text{Godunov for } L) \equiv \text{FD for } (\text{ODE}) \]

Godunov for \( L \) \( \equiv \) FD for \( (\text{ODE}) \)

**THM:** Since \( w \) is constant on each cell as for Riemann Pb, total variation in \( x \) does not increase in time.
THEOREM: Moreover, \( 0 \leq N \leq N_{\text{max}}, \quad 0 \leq \rho = \frac{1}{c} \leq \rho_{\text{max}} = 1 \).

(Compare with PW !)

If \( (\Delta X, \Delta t) \to 0 \) with \( \frac{\Delta t}{\Delta X} = C \) and CFL, \((\text{God})\) CV to \((L)\)

Rigorous statement (!)

If \( \Delta X \) fixed and \( \Delta t \to 0 \), \((\text{God}) = (\text{FD})\) CV to \((\text{ODE}') = (\text{ODE})\)

AND \((\text{ODE}')\) INHERITS THE BV ESTIMATES (NO OSCILLATION)

FROM GODUNOV.

Then when \( \Delta t \to 0 \), \((\text{ODE}')\), which is exactly the \( \frac{1}{2} \) discretization of \((L)\), CV to the (unique) weak entropy solution of \((L)\).

The two limits are the same: commutation of limits

Finally, even for solutions with shocks,

Lagrangian system \((L) \iff \text{Eulerian system} \ (E)\)

(lazy from vacuum)
Eulerian macroscopic view:

\[ \begin{align*}
\frac{\partial t}{\partial t} + \frac{\partial}{\partial x}(\rho w) &= 0 \\
\frac{\partial w}{\partial t} + \rho \frac{\partial w}{\partial x} &= 0
\end{align*} \]

\[ w = u + \rho(\rho) = u + p(T), \quad T = \frac{1}{\rho} \]

\[ \rho(\rho) = u_{\text{max}} - V_{\text{eq}}(\rho) \]

much more general, contains 1st order models \((w=0)\),

\[ w = \text{lagrangian marker} \] (e.g., affecting trucks...)

Lagrangian mass coordinates: Courant-Friedrichs

\[ \begin{align*}
\frac{\partial t}{\partial t} &= \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right) \\
\frac{\partial X}{\partial t} &= S_X^1 \left( \frac{\partial T}{\partial x} \right)
\end{align*} \]

\((x,t) \rightarrow (X,T;t): \begin{align*}
\frac{\partial}{\partial t} &= \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} &= 0 + \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right)
\end{align*} \]

\[ \frac{\partial t}{\partial t} + \frac{\partial x}{\partial t} \rho + \frac{\partial x}{\partial t} w = 0 = \left( \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} \right) \Rightarrow \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial x} w = 0
\]

\[ \mathbf{L} \Rightarrow \mathbf{E}, \text{ even for weak solutions} \]

\[ \begin{align*}
(\text{God}) \iff \mathbf{E} \iff \mathbf{L}, \text{ even for weak solutions}
\end{align*} \]

\[ \theta + \frac{1}{2} = \theta + \frac{1}{2} + \Delta t \frac{\partial w}{\partial x} \]

\[ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial x} \frac{\partial w}{\partial x} = 0 \]

\[ \frac{\partial w}{\partial x} = 0 \]

Link with Hamilton-Jacobi...
24. Case with relaxation term ... \( \partial_t \rho + \partial_x (\rho u) = 0 \), \( w = u + p \rho \) \( w = u + p \rho \) \[ (10) \]

If \( A \neq 0 \), Eulerian system becomes \( \partial_t (\rho w) + \partial_x (\rho w u) = \frac{A}{T_n} \rho (V_{eq}(\rho) - v) \), \( V_{eq}(\rho) := \partial_{\rho} \left( \frac{1}{\rho} \right) \)

Whitham sub-characteristic condition: "convective term must dominate relaxation term"

This condition is necessary for CV of the zero-relaxation limit, !!

Here, in the scaling \((x,t) \rightarrow (x', t') = (x/t', t')\), the relaxation term becomes \( \frac{A}{\varepsilon T_n} \rho (V_{eq}(\rho) - v) \Rightarrow \) (Formally) CV to \( v = V_{eq}(\rho) \) \( \partial_t \rho + \partial_x (\rho V_{eq}(\rho)) = 0 \) LWR

(Unless \( A \rightarrow E \lambda \Rightarrow A \) scale \( T_n \) independent).

Here, sub-character condition: \[ -\rho'(\rho) \leq V_{eq}(\rho) \leq 0 \]

\( \rho > 0 \)

Chapman-Enskog expansion:
\( v = V_{eq}(\rho) + \varepsilon u + ... \)
\( \partial_t u + \partial_x (\rho V_{eq}(\rho)) = \varepsilon \partial_x \left( \rho V_{eq}(\rho) (V_{eq}(\rho) + \rho'(\rho)) \partial_x \rho \right) + ... \)

Possible wild instabilities (Hadamard) if wrong sign. Formal argument don't work in this case!! when \( \varepsilon \rightarrow 0 \) .

Example: C.LL \( \partial_t u^e + \partial_x u^e = 0 \)
\( \partial_t w^e + \partial_x w^e = 0 \)
\( \partial_t (u^e - w^e) + \partial_x (u^e - w^e) = \frac{A}{\varepsilon} (u^e - w^e) \)

Formal limit: \( \partial_t u + \partial_x u = 0 \)
\( \rightarrow \Delta_{xt} u = 0 \) !!

Now, for fixed \( \varepsilon \) see oral discussion.