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Microscopic Models under a Macroscopic View

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Outline:

- General
- Dynamics of the microscopic model (homogeneous case)
- Dynamics of the microscopic model (non-homogeneous case and roadworks)
- Macroscopic view
- Fundamental diagrams
- Future

Basic concept: Take a very simple microscopic model (Bando), study the full dynamics, take a macroscopic view on the results.
Microscopic Bando model on a circular road (scaled)

$N$ cars on a circular road of length $L$:

Behaviour: $x_j$ position of the $j$-th car

$$
\ddot{x}_j(t) = -\left\{ V \left( x_{j+1}(t) - x_j(t) \right) - \dot{x}_j(t) \right\}, \quad j = 1, \ldots, N, \quad x_{N+1} = x_1 + L
$$

$V = V(x)$ optimal velocity function:

$V(0) = 0$, $V$ strictly monotonically increasing, $\lim_{x \to \infty} V(x) = V_{max}$
System for the headways: \( y_j = x_{j+1} - x_j \)

\[
\begin{align*}
\dot{y}_j &= z_j \\
\dot{z}_j &= -\left\{V(y_{j+1}) - V(y_j) - \dot{z}_j\right\}, \quad j = 1, \ldots, N, \quad y_{N+1} = y_1
\end{align*}
\]

Additional condition: \( \sum_{j=1}^{N} y_j = L \)

“quasistationary” solutions: \( y_{s;j} = \frac{L}{N}, \quad z_{s;j} = 0, \quad j = 1, \ldots, N. \)

Linear stability-analysis around this solution gives for the Eigenvalues \( \lambda \):

\[
(\lambda^2 + \lambda + \beta)^N - \beta^N = 0, \quad \beta = V'(\frac{L}{N})
\]

Result (Huijberts (‘02)):

For \( \frac{1}{1 + \cos \frac{2\pi}{N}} > \beta^{max} = \max_x V'(x) \) asymptotic stability

For \( \frac{1}{1 + \cos \frac{2\pi}{N}} = V'(\frac{L}{N}) \) loss of stability
What kind of loss of stability? (I.G., G.Sirito, B. Werner '04):

Eigenvalues as functions of $\beta = V'(\frac{L}{N})$

Bifurcation analysis gives a Hopf bifurcation.

Therefore we have locally periodic solutions.

Are these solutions stable? (i.e. is the bifurcation sub- or super-critical?)
Criterion: Sign of the first Ljapunov-coefficient $l$

Theorem:

$$l = c^2 \left\{ V''' \left( \frac{L}{N} \right) - \frac{\left( V'' \left( \frac{L}{N} \right) \right)^2}{V' \left( \frac{L}{N} \right)} \right\}$$

Conclusion: For the mostly used (Bando et al (95))

$$V(x) = V_{\text{max}} \frac{\tanh(a(x - 1)) + \tanh a}{1 + \tanh a}$$

the bifurcation is supercritical (i.e. stable periodic orbits).

But: “Similar” functions $V$ give also subcritical bifurcations.
Problem: It seems to be very sensitive with respect to $V$

Global bifurcation analysis: numerical tool (AUTO2000)

Conclusion: Globally “similar” functions $V$ give similar behaviour. The bifurcation is “macroscopically” subcritical

Conclusion for the application: the critical parameters from the linear analysis are not relevant
More bifurcations:

Eigenvalues as functions of $\beta = V'(L/N)$

Conclusion: There are many other (weakly unstable) periodic solutions
(J. Greenberg '04,'07) Solutions with many oscillations finally tend to a solution with one oscillation

(G. Oroz, R.E. Wilson, B. Krauskopf '04, '05) Qualitatively the same global bifurcation diagram for the model with delay
Extension to “standart” microscopic model

Every driver is “aggressive” with weight $\alpha$

$$\ddot{x}_j(t) = -\frac{1 - \alpha}{\tau} \left\{ V \left( x_{j+1}(t) - x_j(t) \right) - \dot{x}_j(t) \right\} + \alpha \left\{ \dot{x}_{j+1}(t) - \dot{x}_j(t) \right\},$$

$j = 1, \ldots, N, \quad x_{N+1} = x_1 + L$ optimal velocity-function:

loss of stability similar

“Aggressive” drivers stabilize the traffic flow!

unfortunately also the number of accidents increases!

(Olmos & Munos, Condensed matter 2004)
Rasert hegen Stabben Frei

ten Stier - doch die Gegen von Straßensicht,
aggressivem Manager: Die Zahl der Verkehrste-
jene in Bogota wie Rallye-Fahrer. Die Folge der
fahrt in New York: der Londoner Verhältnissich
kehlLasse simuliert, im Vergleich zu den Auto-
chen Computerprogramm erstellt, das den Ver-
ten Ihrer Landesart ausgeschichtet und daraus
sehen. Nationaluniversität haben das Fabrikmal-
chen, die wissen haben, der Kultur-Darstellung
der Reichsakademie. Fabriken der Kultur-Darstellung
Autors, nicht in Straßensicht, soll ausgeschichtet
nen Einwohner und macht die einiger Million
Dass Bogota, eine Stadt mit zwei Straßen Millio-

"DIE Zeit" 18.6.04
Symmetry breaking, the above theory is not easily applicable

A solution is called ponies on a Merry-Go-Round solution (short POM), if there is a $T \in \mathbb{R}$, such that

(i) $x_i(t + T) = x_i(t) + L$ \hspace{1em} (i = 1, \ldots, N)

(ii) $x_i(t) = x_{i-1} \left( t + \frac{T}{N} \right)$ \hspace{1em} (i = 1, \ldots, N)

hold (Aronson, Golubitsky, Mallet-Paret '91). We call $T$ rotation number and $\frac{T}{N}$ the phase (phase shift).
**Theorem:** The above model has POM solutions for small $\epsilon > 0$.

Velocity of the quasistationary solution (no roadwork) versus roadwork solution (The red line indicates maximum velocity).
Technique: Poincare maps

\[ \Pi(\eta) = \Phi_{T(\eta)}(\eta) - \Lambda, \] where \( \Phi \) is the induced flow and \( \Lambda \)
reduces the spacial components by \( L \).

Study fixed points of the corresponding Poincare and reduced
Poincare maps. Roadworks are (regular) perturbations.
A curve of Neimark-Sacker bifurcations in the $(L, \epsilon)$-plane for $N = 5$. 
Four different attractors:

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon = 0$</td>
<td>trivial POM $x^0$</td>
<td>Hopf periodic solution</td>
</tr>
<tr>
<td>$\epsilon &gt; 0$</td>
<td>POM $x^\epsilon$</td>
<td>quasi-POM</td>
</tr>
</tbody>
</table>

i.e. here

POM’s are typically perturbed quasistationary solutions

quasi-POM’s are perturbed (Hopf) periodic solutions
Invariant curves:

Two closed invariant curves ($\epsilon = 0$ and $\epsilon > 0$) of the reduced Poincaré map $\pi$. On the left also the optimal velocity function $V_0$ is given in gray.
The 4 different scenarios:

above: no roadworks, below: with roadworks
Macroscopic view of the 4 different scenarios:

above: no roadworks, below: with roadworks
Macroscopic view of density and velocity:

strong road work influence ($\epsilon = 0.32$)
A real world point of view on the reduced Poincaré map $\pi$ for $N = 10, \epsilon = 0$. 

Fundamental diagrams:
Fundamental diagrams I:

Overlapped fundamental diagrams for $N = 10, L = 50, \ldots, 4$ measuring at a fixed point.
Fundamental diagrams II:

Fundamental diagram of time-averaged flow versus average density for $N = 10, L = 50\ldots4$. 
Fundamental diagrams III (with roadworks):

Overlapped fundamental diagrams for $N = 10, L = 50, ..., 4, \quad \epsilon = 0.1$ measuring at a fixed point.
Current and future work:

- Is this dynamics contained in macroscopic models?
- Which macroscopic model has the same (rich) dynamics than the basic Bando model
- Micro-macro link (Aw, Klar, Materne, Rascle 2002)