Remarks on the blowup criteria for 3D Navier-Stokes equations: critical vs. non-critical norms

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KE: Navier-Stokes equations, Leray equations, blowup

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Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \triangle \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

with smooth initial data with finite energy in $\mathbb{R}^3$

J. Leray, Acta Math. **63**, 193 (1934)

Plan

1. Introduction: review of blowup criteria

2. Asymptotic analysis

3. Blowup criteria in L^∞-norms

5. Summary

(4. Late singularity for type II singularity)

1. Introduction: review of blowup criteria

Scale-invariance and criticality $\boldsymbol{x} \to \boldsymbol{x}/\lambda, t \to t/\lambda^2$

$$\boldsymbol{u}(\boldsymbol{x}, t) \to \boldsymbol{U}(\boldsymbol{\xi}, \tau) \equiv \lambda^{-1}\boldsymbol{u}(\boldsymbol{x}/\lambda, t/\lambda^2)$$

$$p(\boldsymbol{x}, t) \to P(\boldsymbol{\xi}, \tau) \equiv \lambda^{-2}p(\boldsymbol{x}/\lambda, t/\lambda^2)$$

example: $n =$ spatial dimension

$$\boldsymbol{x} = \lambda\boldsymbol{\xi}, \boldsymbol{t} = \lambda^2\boldsymbol{\tau}, \boldsymbol{u} = \lambda^{-1}\boldsymbol{U}$$

$$\int |\boldsymbol{u}|^q d\boldsymbol{x} = \lambda^{n-q} \int |\boldsymbol{U}|^q d\boldsymbol{\xi},$$

$q < n :$ super-critical, $q = n :$ critical, $q > n :$ sub-critical

$$\int_{\mathbb{R}^4} |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} = [\nu^2], \ \int_{\mathbb{R}^2} |\boldsymbol{u}|^2 d\boldsymbol{x} = [\nu^2]$$

Blowup criteria (subcritical)

Leray bounds

$$\int_{\mathbb{R}^3} |\boldsymbol{\omega}|^2 d\boldsymbol{x} \geq C \frac{\nu^{3/2}}{\sqrt{t_* - t}}$$

$$\sup_{\boldsymbol{x}} |\boldsymbol{u}(\boldsymbol{x}, t)| \geq c \frac{\nu^{1/2}}{\sqrt{t_* - t}}$$

Notations

energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \frac{1}{2}\|\boldsymbol{u}\|^2_{L^2(\mathbb{R}^n)}$$

enstrophy

$$Q(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \frac{1}{2}\|\boldsymbol{u}\|^2_{H^1(\mathbb{R}^n)}$$

$H^{1/2}$-norm, defined with $\Lambda \equiv (-\triangle)^{1/2} \leftrightarrow |\boldsymbol{k}|$

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\Lambda^{1/2}\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \frac{1}{2}\|\boldsymbol{u}\|^2_{H^{1/2}(\mathbb{R}^n)}$$

Leray's self-similar blowup ansatz (1934)

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{[2a(t_* - t)]^{1/2}} \boldsymbol{U}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} = \frac{\boldsymbol{x}}{[2a(t_* - t)]^{1/2}}$$

Leray equations

$$\boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + a(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + \boldsymbol{U}) = -\nabla_{\boldsymbol{\xi}} P + \nu \triangle_{\boldsymbol{\xi}} \boldsymbol{U},$$

$$\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{U} = 0$$

No go theorems: self-similar blowup ruled out
If $\boldsymbol{U} \in L^3(\mathbb{R}^3)$ then $\boldsymbol{U} \equiv 0$ Nečas, Růžička and Šverák (1996), also Tsai (1998) for $\boldsymbol{U} \in L^q(\mathbb{R}^3)$, $q > 3$.

Blowup criteria (critical)

$$\|\boldsymbol{u}\|_{L^3(\mathbb{R}^3)} \to \infty \ \text{ as } \ t \to t_*$$

Escauriaza, Seregin & Sverak (2003). By $\|\boldsymbol{u}\|_{L^3} \leq \|\boldsymbol{u}\|_{H^{1/2}}$

$$\|\boldsymbol{u}\|_{H^{1/2}(\mathbb{R}^3)} \to \infty \ \text{ as } \ t \to t_*$$

$$\int |\Lambda^{1/2}\boldsymbol{u}|^2 d\boldsymbol{x} = \int \boldsymbol{u} \cdot \Lambda \boldsymbol{u} d\boldsymbol{x} \leq \left(\int |\boldsymbol{u}|^2 d\boldsymbol{x} \int |\boldsymbol{\omega}|^2 d\boldsymbol{x} \right)^{1/2}$$

$$EQ = \text{surrogate for } H^{1/2}$$

$$\frac{1}{2}|\Lambda^{1/2}\boldsymbol{u}|^2 \leq \boldsymbol{u} \cdot \Lambda \boldsymbol{u} \ \text{ on } \mathbb{R}^3$$

Cordoba-Cordoba(2003)

Non-steady Leray equations

dynamics rescaling

Assuming N-S solutions blows up at $t = t_*$ apply dynamics rescaling.

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{\sqrt{2a(t_* - t)}}\boldsymbol{U}(\boldsymbol{\xi},\tau)$$

$$\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\sqrt{2a(t_* - t)}},\ \tau = \int_0^t \frac{ds}{\lambda(s)^2} = \frac{1}{2a}\log\frac{t_*}{t_* - t}$$

$$\left(t = \frac{1 - e^{-2a\tau}}{2a},\ \ t_* = \frac{1}{2a},\ \ \lambda(t) = \sqrt{2a(t_* - t)}\right)$$

non-stationary Leray equations

$$\frac{\partial \boldsymbol{U}}{\partial \tau} + \boldsymbol{U}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U} + a(\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U} + \boldsymbol{U}) = -\nabla_{\boldsymbol{\xi}}P + \nu\triangle_{\boldsymbol{\xi}}\boldsymbol{U},\ \ \nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{U} = 0$$

No go theorems: Chae (2007), Hou and Li (2007)

Asymptotic self-similar blowup ruled out

$$\lim_{\tau \to \infty} \|\boldsymbol{U}(\boldsymbol{\xi}, \tau) - \overline{\boldsymbol{U}}(\boldsymbol{\xi})\|_{L^p} = 0, \ \overline{\boldsymbol{U}} \in L^p, \ p \geq n$$

then $\overline{\boldsymbol{U}}$ is a steady solution to the Leray equations.
In fact, $\overline{\boldsymbol{U}} \equiv 0$ by NRS(1996) and Tsai(1998)

3D Navier-Stokes

$$\Longrightarrow \quad \frac{dQ}{dt} \leq C\frac{Q^3}{\nu^3} \quad \Longrightarrow \quad \int_0^{t_*} Q(t')^2 dt' = \infty$$

$$\searrow Q(t) \geq c\frac{\nu^{3/2}}{\sqrt{t_* - t}} \quad \Longrightarrow \quad T \simeq \frac{\nu^3}{Q(0)^2}$$

3D Euler

$$\frac{d\|\boldsymbol{\omega}\|_\infty}{dt} \leq C\|\boldsymbol{\omega}\|_\infty^2 \ \text{ invalid} \qquad \searrow \int_0^{t_*} \|\boldsymbol{\omega}(t')\|_\infty dt' = \infty \ \text{ BKM}$$

$$\Longrightarrow \quad \|\boldsymbol{\omega}\|_\infty = O\left(\frac{1}{t_* - t}\right)$$

cf. $\dfrac{d\|\boldsymbol{\omega}\|_\mu}{dt} \le C\|\boldsymbol{\omega}\|_\mu^2$ Frisch & Bardos(1975)

$$\|\boldsymbol{\omega}(t)\|_\mu \le \frac{\|\boldsymbol{\omega}_0\|_\mu}{1 - Ct\|\boldsymbol{\omega}_0\|_\mu}, \quad T \simeq \frac{1}{\|\boldsymbol{\omega}_0\|_\mu}$$

$$\|\boldsymbol{\omega}\|_\mu \equiv \|\boldsymbol{\omega}\|_\infty + \sup_{\boldsymbol{x}\neq\boldsymbol{y}} \frac{|\boldsymbol{\omega}(\boldsymbol{x}) - \boldsymbol{\omega}(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^\mu}, \quad \mu > 0$$

not scaling-friendly

3D Navier-Stokes

$$\frac{d^2\|\boldsymbol{u}\|_{L^3}}{dt^2} \leq \frac{C}{\nu^3}\left(\frac{d\|\boldsymbol{u}\|_{L^3}}{dt}\right)^2 \quad ?$$

$$\searrow \|\boldsymbol{u}\|_{L^3} = \int_0^t \frac{d\|\boldsymbol{u}\|_{L^3}}{dt'}dt' \to \infty \text{ as } t \to t_*, \text{ ESS(2007)}$$

$$\Longrightarrow \frac{d\|\boldsymbol{u}\|_{L^3}}{dt} = O\left(\frac{\nu}{t_* - t}\right) \Longrightarrow \|\boldsymbol{u}\|_{L^3} = O\left(\nu \log \frac{1}{t_* - t}\right)$$

$$\text{Discard } \frac{d\|\boldsymbol{U}\|_{L^3}}{d\tau} = O\left(\frac{1}{\tau}\right) \text{ by Chae, Hou-Li(2006)}$$

associated with

$$\|\boldsymbol{U}\|_{L^3} = \int_0^\tau \frac{d\|\boldsymbol{U}\|_{L^3}}{d\tau'}d\tau' \to \infty \text{ as } \tau \to \infty$$

Critical case

$$\frac{\partial \boldsymbol{U}}{\partial \tau} + \boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + a(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + \boldsymbol{U}) = -\nabla_{\boldsymbol{\xi}} P + \nu \triangle_{\boldsymbol{\xi}} \boldsymbol{U}$$

The equations for L^n-norm are *identical* before/after rescaling

$$\frac{1}{n}\frac{d}{dt}\int_{\mathbb{R}^n} |\boldsymbol{u}|^n d\boldsymbol{x} = -\int_{\mathbb{R}^n} |\boldsymbol{u}|^{n-2} \nabla \cdot (\boldsymbol{u} p) d\boldsymbol{x} + \nu \int_{\mathbb{R}^n} |\boldsymbol{u}|^{n-2} \boldsymbol{u} \cdot \triangle \boldsymbol{u} d\boldsymbol{x}$$

$$\frac{1}{n}\frac{d}{d\tau}\int_{\mathbb{R}^n} |\boldsymbol{U}|^n d\boldsymbol{\xi} = -\int_{\mathbb{R}^n} |\boldsymbol{U}|^{n-2} \nabla_{\boldsymbol{\xi}} \cdot (\boldsymbol{U} P) d\boldsymbol{\xi} + \nu \int_{\mathbb{R}^n} |\boldsymbol{U}|^{n-2} \boldsymbol{U} \cdot \triangle_{\boldsymbol{\xi}} \boldsymbol{U} d\boldsymbol{\xi}$$

by orthogonality

$$a \int_{\mathbb{R}^n} |\boldsymbol{U}|^{n-2} \boldsymbol{U} \cdot (\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + \boldsymbol{U}) d\boldsymbol{\xi} = 0$$

Impossible to distinguish: $\boldsymbol{u}$ (blow-up), $\boldsymbol{U}$ (grow-up)

Critical & subcritical norms

norms	time of local exist.	small data $\Rightarrow$ global exist.
L^n	NA	Yes
L^p $(p > n)$	$\|\boldsymbol{u}_0\|^{-2p/(p-n)}$	NA

2. Asymptotic analysis

Enstrophy equation

$$\frac{dQ}{dt} = \int_{\mathbb{R}^3} \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\omega} d\boldsymbol{x} - \nu \int_{\mathbb{R}^3} |\nabla \times \boldsymbol{\omega}|^2$$

well-known bound

$$\frac{dQ}{dt} \leq C\frac{Q^3}{\nu^3} - \frac{5\nu}{4}\frac{Q^2}{E}$$

With $\dfrac{dE}{dt} = -2\nu Q, \quad \dfrac{d}{dt}\log(EQ) \leq C\dfrac{Q^2}{\nu^3}$

$$E(t)Q(t) \leq E(0)Q(0)\exp\left(\frac{C}{\nu^3}\int_0^t Q(t')^2 dt'\right)$$

Type I: $Q(t) \approx \dfrac{\nu^{3/2}}{\sqrt{t_* - t}} \Rightarrow$ a power-law upperbound

$$\times \frac{1}{C\nu^5} \frac{E^2}{Q}$$

$$\frac{dQ}{dt} \leq C\frac{Q^3}{\nu^3} - \frac{5\nu}{4}\frac{Q^2}{E}$$

$\Longleftrightarrow$

$$\frac{E^2}{C\nu^5} \frac{d \log Q}{dt} \leq f^2 - \frac{5}{4C} f, \quad f(t) \equiv \frac{E(t)Q(t)}{\nu^4}$$

$\Longleftrightarrow$

$$f \geq \left(\frac{5}{4C} + \sqrt{\left(\frac{5}{4C}\right)^2 + \frac{4E^2}{C\nu^5} \frac{d \log Q}{dt}} \right)$$

Essentially

$$f \gtrsim \frac{E}{\nu^{5/2}} \sqrt{\frac{d}{dt} \log Q}$$

$$\frac{1}{2}\|\boldsymbol{u}(t)\|^2_{H^{1/2}} \equiv H(t) \leq \sqrt{E(t)Q(t)}$$

i) $H(t) \simeq \sqrt{E(t)Q(t)}$
ii) $H(t) \ll \sqrt{E(t)Q(t)}$
We consider case i)

Robinson, Sadowski & Silva (2012)

$$\|\boldsymbol{u}(t)\|_{H^{1/2}} \leq C \log \frac{t_*}{t_* - t} + \|\boldsymbol{u}(0)\|_{H^{1/2}}$$

$$\sqrt{E(t)Q(t)} \leq C \left(\log \frac{t_*}{t_* - t} \right)^2$$

A simple analysis

Assume f behaves in a tame fashion: e.g. $f(t) \leq' \log \frac{1}{t - t_*}'$

$$\frac{dQ}{dt} \leq C\nu^5 \frac{Q}{E^2} \times \text{'log'}$$

$$\begin{cases} \dfrac{dQ}{dt} \lesssim C\nu^5 \dfrac{Q}{E^2}, \\ \dfrac{dE}{dt} = -2\nu Q \end{cases}$$

Derivation

$$\ddot{e} \geq \frac{\dot{e}}{e^2} \equiv F(e, \dot{e})$$

$W \equiv \dot{e}$

$$\ddot{e} = \frac{dW}{dt} = \frac{dW}{de}\dot{e}$$

$$W\frac{dW}{de} \geq F(e, \dot{e})$$

$$\frac{dW}{de} \leq \frac{F(e, \dot{e})}{W} = \frac{1}{e^2}$$

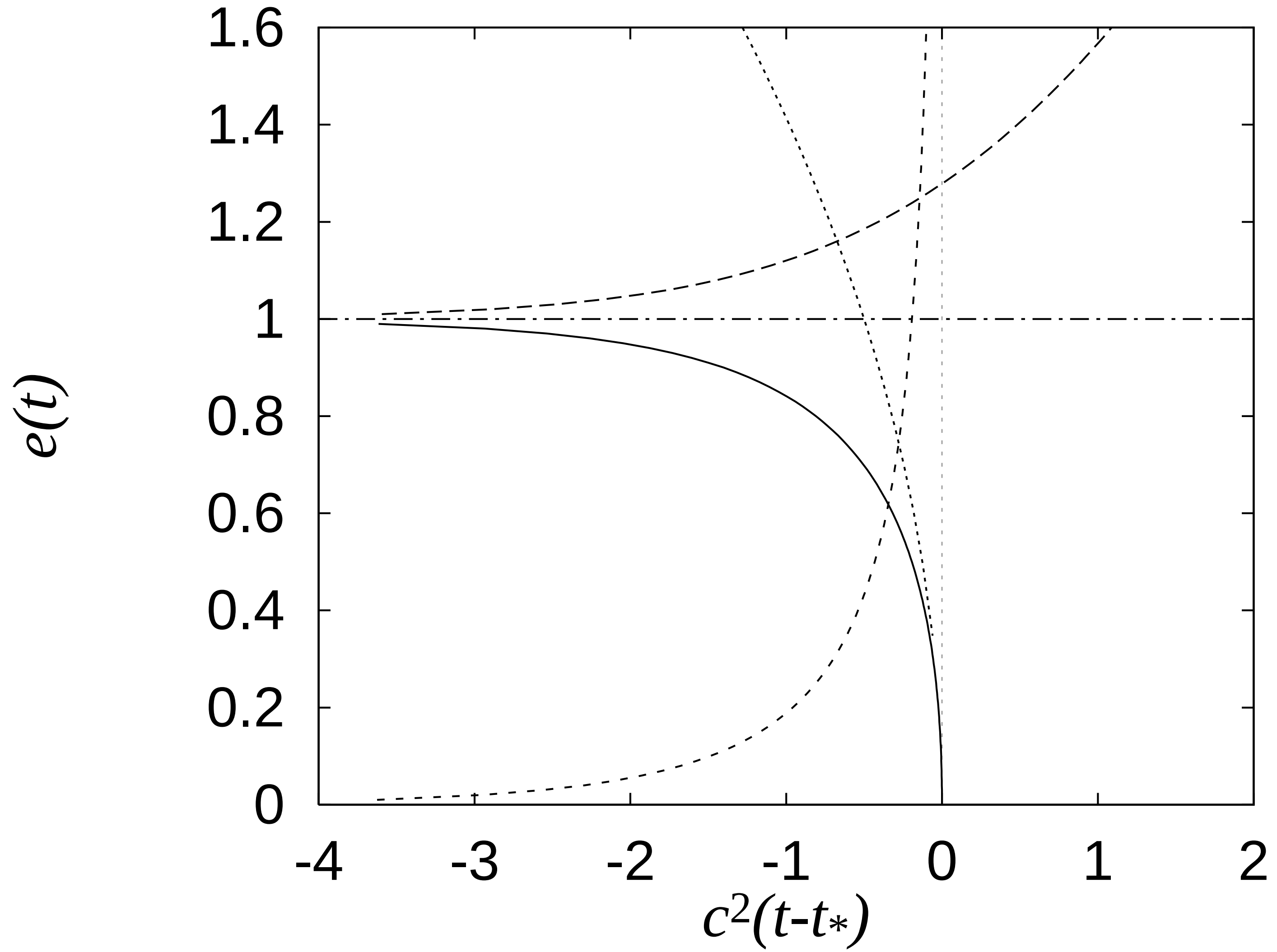
By $e(t) \equiv \frac{E(t)}{C^{1/2}\nu^{5/2}}$

$$\log|1 - c\,e(t)| + c\,e(t) \gtrsim c^2\,(t_* - t)$$

Taylor expansion: $e(t) \lesssim \sqrt{2(t_* - t)}$

$q(t) \equiv -\dot{e}(t)$, $e(t) \simeq \frac{1}{q(t)+c}$

$$\log\left(\frac{q(t)}{q(t)+c}\right) + \frac{c}{q(t)+c} \simeq c^2(t_* - t)$$



1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
e(t)
-4
-3
-2
-1
0
1
2
c2(t-t*)

Numerical experiments

pseudo-spectral

$$N = 256, \nu = 1 \times 10^{-3}, \Delta t = 2 \times 10^{-3}$$

Case 1: random initial data

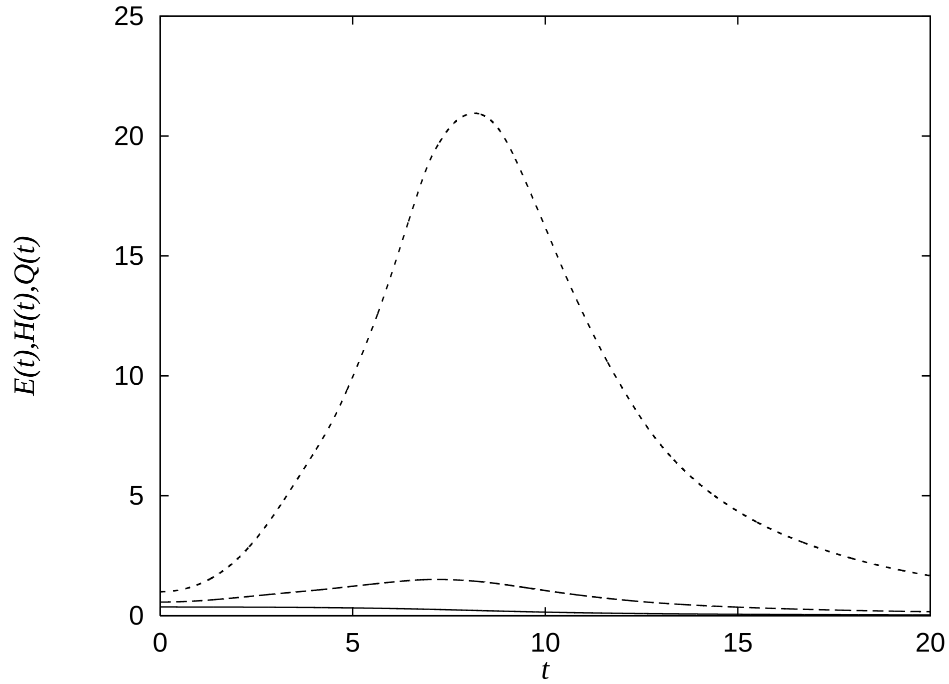
$$E(k) \propto k^4 \exp(-k^2)$$

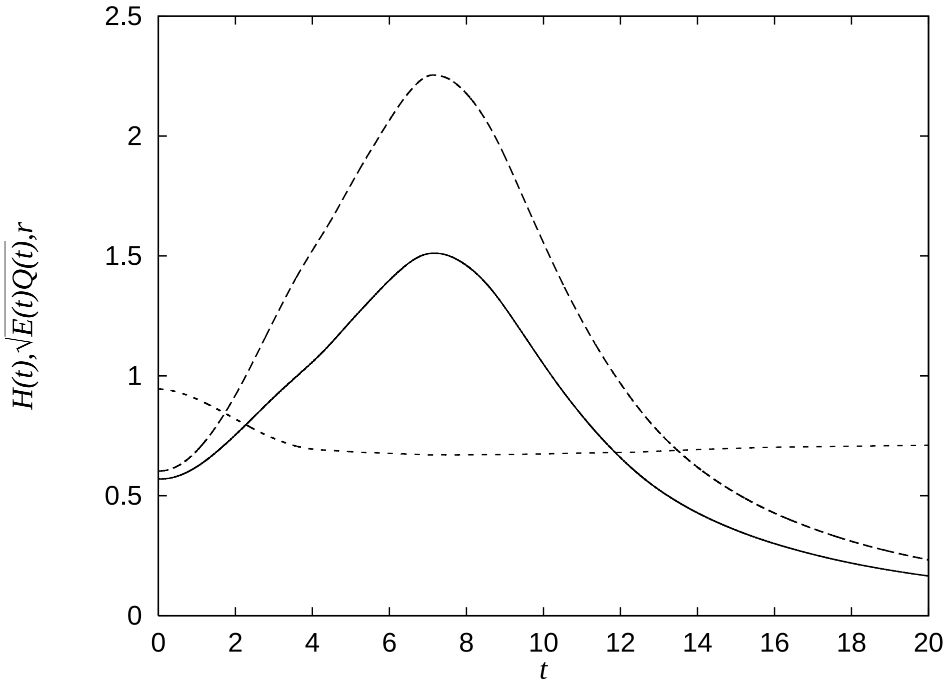
Case 2: the Taylor-Green vortex

$$\boldsymbol{u} = \begin{pmatrix} \cos x \sin y \sin z \\ -\sin x \cos y \sin z \\ 0 \end{pmatrix}$$

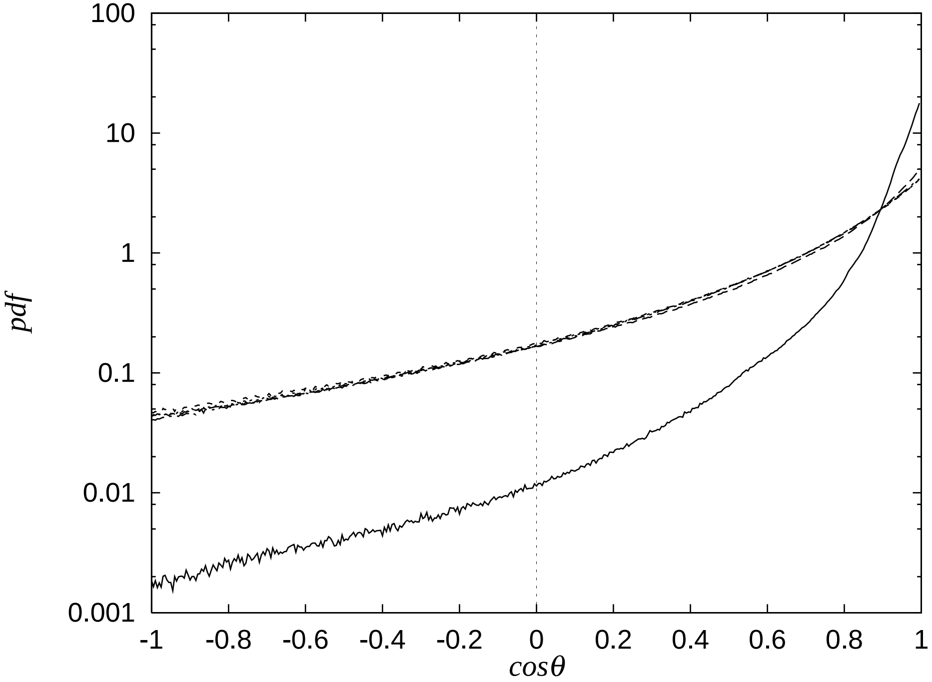
$\Lambda \boldsymbol{u} = \sqrt{3}\boldsymbol{u}$ and $\boldsymbol{u} \cdot \Lambda \boldsymbol{u} = \sqrt{3}|\boldsymbol{u}|^2 \geq 0$

$$r(t) = \frac{H(t)}{\sqrt{E(t)Q(t)}}$$



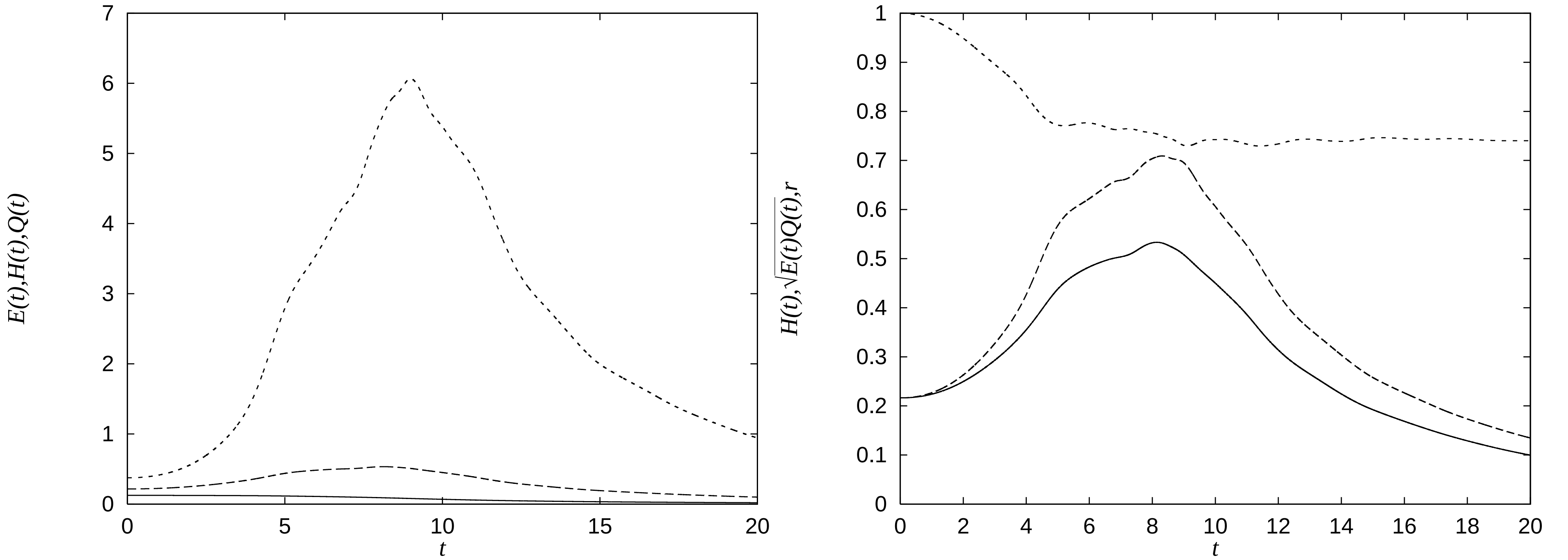


Enstrophy: Case 1

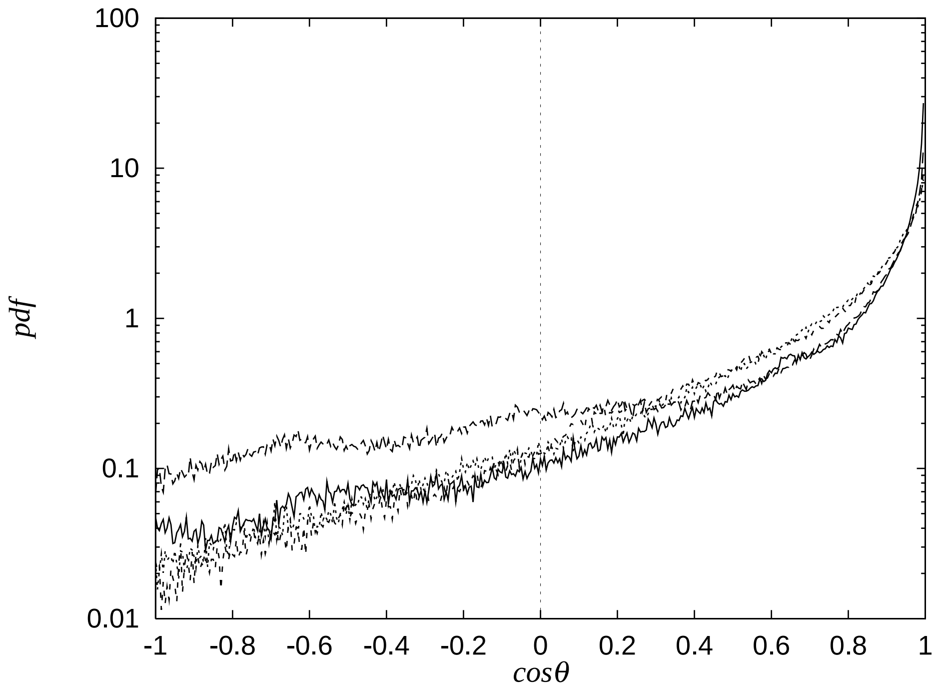


$\cos\theta$ takes negative values,
$r(t) \simeq 0.7$

PDF: Case 1



Enstrophy: Case 2



$\cos\theta$ takes negative values,
$r(t) \simeq 0.7$

PDF: Case 2

3. Blowup criteria in L^∞-norms: $(-\triangle)^\alpha, \;\; \alpha = 0, 1/2, 1$

$$\left(\int |\Lambda^\beta \boldsymbol{A}|^p \mathrm{d}\boldsymbol{x}\right)^{1/p} = \nu L^{3/p-(2\alpha+\beta-2)}$$

For N-S $(-\triangle)^1$: $\;\alpha = 1, \; \beta = 3/p$

$$\int |\Lambda^{3/p} \boldsymbol{A}|^p \mathrm{d}\boldsymbol{x}$$

$$p = 1: \quad \int |\nabla \boldsymbol{\omega}| \mathrm{d}\boldsymbol{x} \quad \text{Yes}$$

$$p = 2: \quad \|\boldsymbol{u}\|_{H^{1/2}} \quad \text{Yes}$$

$$p = 3: \quad \|\boldsymbol{u}\|_{L^3} \quad \text{Yes}$$

$$p = \infty: \quad \|\boldsymbol{A}\|_{L^\infty} \quad ?$$

$$\|\boldsymbol{u}\|_{L^3} \leq C\|D^{1/2}\boldsymbol{u}\|_{L^2}$$

$$\|\boldsymbol{u}\|_{L^3} \leq C\|D^2\boldsymbol{u}\|_{L^1}$$

$$\|\boldsymbol{A}\|_{L^\infty} \leq C\|\nabla \times \boldsymbol{A}\|_{L^3} \quad \text{invalid}$$

$$\|\boldsymbol{A}\|_{\mathrm{BMO}} \leq C\|\nabla \times \boldsymbol{A}\|_{L^3}$$

For $(-\triangle)^0$: $\alpha = 0,\ \beta = 2 + 3/p$

$$\int |\Lambda^{2+3/p}\boldsymbol{A}|^p \mathrm{d}\boldsymbol{x}$$

$$p = 1: \quad \int |D^3\boldsymbol{\omega}|\mathrm{d}\boldsymbol{x} \quad \text{Yes}$$

$$p = 2: \quad \|\boldsymbol{u}\|_{H^{5/2}} \quad \text{Yes}$$

$$p = 3: \quad \|\nabla\boldsymbol{\omega}\|_{L^3} \quad \text{Yes}$$

$$p = \infty: \quad \|\boldsymbol{\omega}\|_{L^\infty}, \|\boldsymbol{\omega}\|_{\mathrm{BMO}} \quad \text{Yes}$$

For $(-\triangle)^{1/2}$, $\alpha = 1/2$, $\beta = 1 + 3/p$

$$\int |\Lambda^{1+3/p}\boldsymbol{A}|^p \mathrm{d}\boldsymbol{x}$$

$$p = 1: \quad \int |\triangle\boldsymbol{\omega}|\mathrm{d}\boldsymbol{x} \quad ?$$

$$p = 2: \quad \|\boldsymbol{u}\|_{H^{3/2}} \quad ?$$

$$p = 3: \quad \|\boldsymbol{\omega}\|_{L^3} \quad ?$$

$$p = \infty: \quad \|\boldsymbol{u}\|_{L^\infty} \quad ?$$

$*\alpha = 1$

Navier-Stokes equations written in vector potentials $\boldsymbol{A}$; ($\boldsymbol{u} = \nabla \times \boldsymbol{A}$)

$$\frac{\partial \boldsymbol{A}}{\partial t} = \frac{3}{4\pi}\mathrm{P.V.}\int_{\mathbb{R}^3} \frac{\boldsymbol{r} \times (\nabla \times \boldsymbol{A}(\boldsymbol{x}'))\, \boldsymbol{r} \cdot (\nabla \times \boldsymbol{A}(\boldsymbol{x}'))}{|\boldsymbol{r}|^5}\, \mathrm{d}\boldsymbol{x}' + \nu \triangle \boldsymbol{A},$$

$$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y},\ \nabla \cdot \boldsymbol{A} = 0$$

$$\text{by} \quad \sup_{\boldsymbol{x}} |\boldsymbol{u}(\boldsymbol{x}, t)| \geq c\frac{\nu^{1/2}}{\sqrt{t_* - t}}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} \simeq C\frac{\nu}{t_* - t}, \quad \boldsymbol{A} \simeq \nu \log \frac{1}{t_* - t}(?)$$

Or, depletion keeps $\boldsymbol{A}$ bounded ?

Leray equations

$$\frac{\partial \tilde{\boldsymbol{A}}}{\partial \tau} + a\boldsymbol{\xi} \cdot \nabla \tilde{\boldsymbol{A}}$$

$$= \frac{3}{4\pi} \mathrm{P.V.} \int_{\mathbb{R}^3} \frac{(\boldsymbol{\xi} - \boldsymbol{\xi'}) \times (\nabla \times \tilde{\boldsymbol{A}}(\boldsymbol{\xi}')) \, (\boldsymbol{\xi} - \boldsymbol{\xi}') \cdot (\nabla \times \tilde{\boldsymbol{A}}(\boldsymbol{\xi}'))}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^5} \, \mathrm{d}\boldsymbol{\xi}' + \nu \triangle \tilde{\boldsymbol{A}}$$

$$*\alpha = 0 \quad \boldsymbol{\omega} = e^{-\nu t}\tilde{\boldsymbol{\omega}}, \;\; \boldsymbol{u} = e^{-\nu t}\tilde{\boldsymbol{u}}, \;\; t' = \frac{1 - e^{-\nu t}}{\nu}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} - \nu \boldsymbol{\omega}$$

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t'} + \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{\omega}} = \tilde{\boldsymbol{\omega}} \cdot \nabla \tilde{\boldsymbol{u}}, \;\; \|\tilde{\boldsymbol{\omega}}\|_\mu \leq \frac{\|\tilde{\boldsymbol{\omega}}_0\|_\mu}{1 - K\|\tilde{\boldsymbol{\omega}}_0\|_\mu t'}$$

$$T' = \frac{1}{K\|\tilde{\boldsymbol{\omega}}_0\|_\mu}, \;\; T = \frac{1}{\nu} \log \frac{1}{1 - \frac{\nu}{K\|\boldsymbol{\omega}_0\|_\mu}}$$

Note that if $\|\boldsymbol{\omega}_0\|_\mu = \nu/K$, $T = \infty$

$$\text{If} \quad e^{-\nu t_*} = 1 - \frac{\nu}{K\|\boldsymbol{\omega}_0\|_\mu} > 0, \;\; \|\boldsymbol{\omega}\|_\mu \leq \frac{\nu}{K} \frac{1}{1 - e^{\nu(t - t_*)}}$$

Get a feel for $(-\triangle)^{1/2}$ by comparing with $\triangle$

Burgers vortex: $\alpha = 1$

$$\boldsymbol{u} = (-Ar, v(r,t), 2Az)$$

$$\frac{\partial v}{\partial t} - A\left(r\frac{\partial v}{\partial r} + v\right) = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right)$$

$$\frac{\partial \omega}{\partial t} - \frac{A}{r}\frac{\partial}{\partial r}\left(r^2\omega\right) = \nu\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r}\right)$$

$$\omega(r) = \frac{A\Gamma}{2\pi\nu}\exp\left(-\frac{Ar^2}{2\nu}\right)$$

$$u(r) = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{Ar^2}{2\nu}\right)\right]$$

modified Burgers vortex: $\alpha = 1/2$

$$\boldsymbol{u} = (-Ar, v(r,t), 2Az)$$

$$\frac{\partial \omega}{\partial t} - \frac{A}{r}\frac{\partial}{\partial r}\left(r^2\omega\right) = -\nu' \Lambda \omega$$

$$\omega(r) = \frac{\Gamma}{2\pi}\left(\frac{A}{\nu'}\right)^2 \frac{1}{\left[1+\left(\frac{Ar}{\nu'}\right)^2\right]^{3/2}}$$

$$u(r) = \frac{\Gamma}{2\pi r}\frac{\sqrt{1+\left(\frac{Ar}{\nu'}\right)^2}-1}{\sqrt{1+\left(\frac{Ar}{\nu'}\right)^2}}$$

cf Jimenez (1994) hyperviscous case

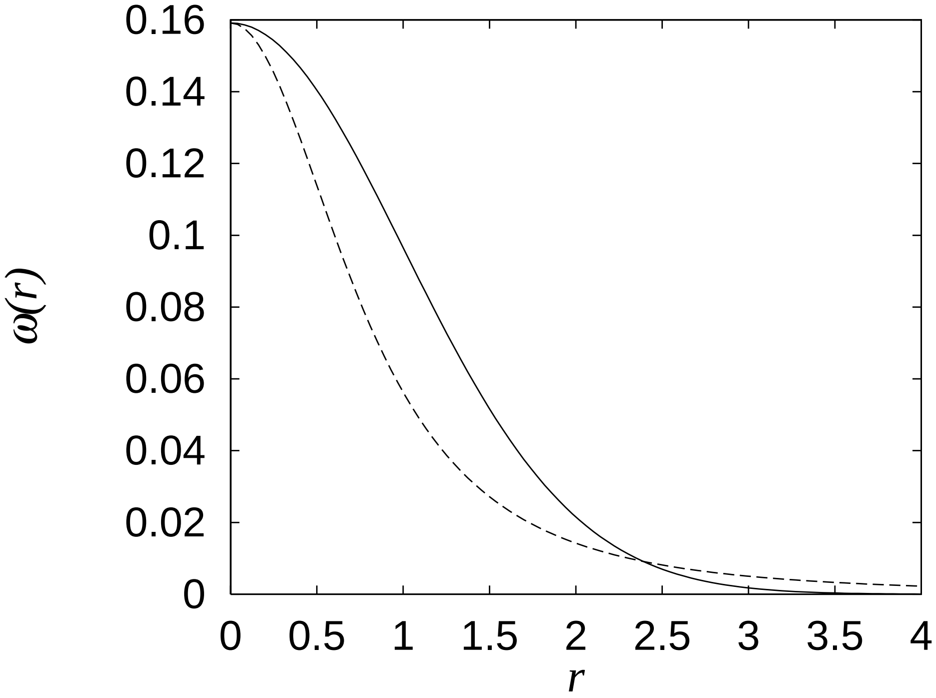
Normalisation

Burgers

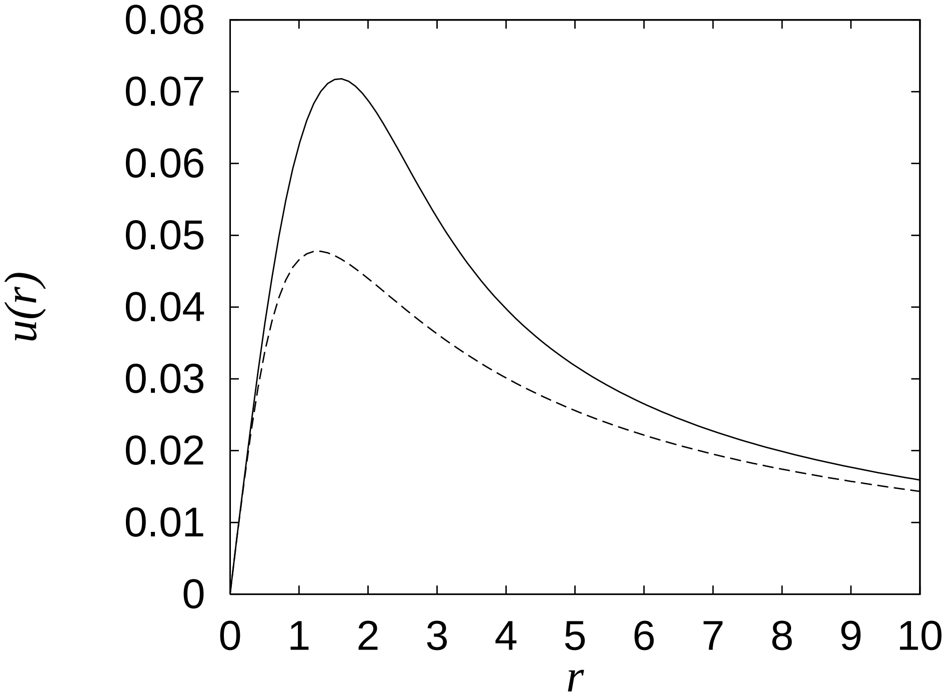
$$\omega(r) = \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right), \quad u(r) = \frac{1}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{2}\right)\right]$$

modified Burgers

$$\omega(r) = \frac{1}{2\pi} \frac{1}{\left(1 + r^2\right)^{3/2}}, \quad u(r) = \frac{1}{2\pi r} \frac{\sqrt{1 + r^2} - 1}{\sqrt{1 + r^2}}$$



0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
$\omega(r)$
0
0.5
1
1.5
2
2.5
3
3.5
4
r



0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
$u(r)$
0
1
2
3
4
5
6
7
8
9
10
r

digression: steady problem is harder

$$\Lambda f(\boldsymbol{x}) = \frac{1}{2\pi}\mathsf{PV}\int_{\mathbb{R}^2} \frac{f(\boldsymbol{x}) - f(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\mathrm{d}\boldsymbol{y}, \quad \Lambda^{-1} f(\boldsymbol{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{f(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y}$$

axis-symmetric case

$$\Lambda f(r) = \frac{2}{\pi}\mathsf{PV}\int_0^\infty \frac{f(r)-f(s)}{(r+s)(r-s)^2} E\left(\frac{2\sqrt{rs}}{r+s}\right) s ds$$

$$\Lambda^{-1} f(r) = \frac{2}{\pi}\int_0^\infty \frac{s f(s)}{r+s} E\left(\frac{2\sqrt{rs}}{r+s}\right) ds, \quad E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 x} dx$$

$$\frac{1}{r}\frac{d}{dr}\left(r^2\omega\right) = \frac{2}{\pi}\mathsf{PV}\int_0^\infty \frac{\omega(r)-\omega(s)}{(r+s)(r-s)^2} E\left(\frac{2\sqrt{rs}}{r+s}\right) s ds$$

5. Summary

- Review of critical/non-critical blowup criteria
- Asymptotic analysis based on $H \simeq \sqrt{EQ}$
- Blowup criteria in L^∞-norms: is $\boldsymbol{A}$ a criterion ?
- For $Q(t) \geq \dfrac{c\nu^{3/2}}{(t_* - t)^{\frac{1}{2}+0}}$, the dangerous time interval scales as $O(R^{4+0})$.