Extreme Vortex States and the Hydrodynamic Blow-Up Problem
(Probing Fundamental Bounds in Hydrodynamics Using Variational Optimization Methods)

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Funded by Early Researcher Award (ERA) and NSERC
Computational Time Provided by SHARCNET
Collaborators

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  *University of Michigan*

- Dmitry Pelinovsky
  *McMaster University*
Agenda

Sharpness of Estimates as Optimization Problem
  Regularity Problem for Navier-Stokes Equation
  Research Program and Earlier Results
  Finite-Time Bounds in 1D Burgers Problem

Bounds for 2D Navier-Stokes Problem
  Bounds on Palinstrophy Growth
  Optimization Problems
  Computational Approach & Results

Bounds for 3D Navier-Stokes Problem
  Bounds on Enstrophy Growth & Optimization Problems
  Extreme Vortex States
  Discussion
Navier-Stokes equation \( (\Omega = [0, L]^d, d = 2, 3) \)

\[
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p - \nu \Delta \mathbf{v} &= 0, & \text{in } \Omega \times (0, T] \\
\nabla \cdot \mathbf{v} &= 0, & \text{in } \Omega \times (0, T] \\
\mathbf{v} &= \mathbf{v}_0 & \text{in } \Omega \text{ at } t = 0 \\
\text{Boundary Condition} & & \text{on } \Gamma \times (0, T]
\end{aligned}
\]

2D Case

- Existence Theory Complete — smooth and unique solutions exist for arbitrary times and arbitrarily large data

3D Case

- Weak solutions (possibly nonsmooth) exist for arbitrary times
- Classical (smooth) solutions (possibly nonsmooth) exist for finite times only
- Possibility of “blow-up” (finite-time singularity formation)
- One of the Clay Institute “Millennium Problems” ($1M!$)
  
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- A Key Quantity — Enstrophy

\[ \mathcal{E}(t) \triangleq \int_{\Omega} |\nabla \times \mathbf{v}|^2 d\Omega (= \|\nabla \mathbf{v}\|_2^2) \]

- Smoothness of Solutions \( \iff \) Bounded Enstrophy
  (Foias & Temam, 1989)

\[ \max_{t \in [0,T]} \mathcal{E}(t) < \infty \]

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- Gronwall’s lemma and energy equation yield \( \forall t \mathcal{E}(t) < \infty \)
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3D Case:
\[
\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3} \mathcal{E}(t)^3
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- corresponding estimate not available ....
- upper bound on \( \mathcal{E}(t) \) blows up in finite time

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\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4 \frac{C\mathcal{E}(0)^2}{\nu^3} t}}
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- singularity in finite time cannot be ruled out!
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Problem of Lu & Doering (2008), I

- Can we actually find solutions which “saturate” a given estimate?
- Estimate $\frac{dE(t)}{dt} \leq cE(t)^3$ at a fixed instant of time $t$

$$\max_{v \in H^1(\Omega), \nabla \cdot v = 0} \frac{dE(t)}{dt}$$

subject to $E(t) = E_0$

where

$$\frac{dE(t)}{dt} = -\nu \| \Delta v \|_2^2 + \int_{\Omega} v \cdot \nabla v \cdot \Delta v \, d\Omega$$

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- Solution using a gradient-based descent method
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Problem of Lu & Doering (2008), II

\[
\left[ \frac{d\mathcal{E}(t)}{dt} \right]_{\text{max}} = 8.97 \times 10^{-4} \mathcal{E}_0^{2.997}
\]
Problem of Lu & Doering (2008), II

\[
\left[ \frac{d\mathcal{E}(t)}{dt} \right]_{\max} = 8.97 \times 10^{-4} \mathcal{E}_0^{2.997}
\]

vorticity field (top branch)
How about solutions which saturate \( \frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^3 \) over a 
\textcolor{red}{finite} time window \([0, T]\)?

\[
\max_{v_0 \in H^1(\Omega), \nabla \cdot v=0} \mathcal{E}(T)
\]
subject to \( \mathcal{E}(0) = \mathcal{E}_0 \)

where

\[
\mathcal{E}(t) = \int_0^t \frac{d\mathcal{E}(\tau)}{d\tau} d\tau + \mathcal{E}_0
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\( \mathcal{E}_0 \) and \( T \) are parameters

In principle doable, but will try something simpler first ...
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## Relevant Estimates

<table>
<thead>
<tr>
<th></th>
<th><strong>Best Estimate</strong></th>
<th><strong>Sharp?</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1D Burgers</strong></td>
<td><strong>instantaneous</strong>  [ \frac{dE(t)}{dt} \leq \frac{3}{2} \left( \frac{1}{\pi^2 \nu} \right)^{1/3} E(t)^{5/3} ]</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td><strong>finite–time</strong>  [ \max_{t \in [0, T]} E(t) \leq \left[ E_0^{1/3} + \left( \frac{L}{4} \right)^2 \left( \frac{1}{\pi^2 \nu} \right)^{4/3} E_0 \right]^3 ]</td>
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<td><strong>2D Navier–Stokes</strong></td>
<td><strong>instantaneous</strong>  [ \frac{dP(t)}{dt} \leq - \left( \frac{\nu}{E} \right) P^2 + C_1 \left( \frac{E}{\nu} \right) P ]</td>
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<td>[ \frac{dP(t)}{dt} \leq \frac{C_2}{\nu} K^{1/2} P^{3/2} ]</td>
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<td><strong>finite–time</strong> [ \max_{t &gt; 0} P(t) \leq \left[ P_0^{1/2} + \frac{C_2}{4\nu^2} K_0^{1/2} E_0 \right]^2 ]</td>
<td><strong>YES</strong> Lu &amp; Doering (2008)</td>
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▸ **Question #1 ("Small")**

Sharpness of *instantaneous* estimates
(at some fixed $t$)

\[
\max_u \frac{d\mathcal{E}}{dt} \quad (1D, 3D)
\]
\[
\max_u \frac{d\mathcal{P}}{dt} \quad (2D)
\]

▸ **Question #2 ("Big")**

Sharpness of *finite–time* estimates
(at some time window $[0, T]$, $T > 0$)

\[
\max_{u_0} \left[ \max_{t\in[0,T]} \mathcal{E}(t) \right] \quad (1D, 3D)
\]
\[
\max_{u_0} \left[ \max_{t\in[0,T]} \mathcal{P}(t) \right] \quad (2D)
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**QUESTION #1 ("SMALL")**

Sharpness of *instantaneous* estimates
(at some *fixed* $t$)

$$\max_u \frac{dE}{dt} (1D, 3D)$$

$$\max_u \frac{dP}{dt} (2D)$$

**QUESTION #2 ("BIG")**

Sharpness of *finite–time* estimates
(at some time window $[0, T]$, $T > 0$)

$$\max_{u_0} \left[ \max_{t \in [0, T]} E(t) \right] (1D, 3D)$$

$$\max_{u_0} \left[ \max_{t \in [0, T]} P(t) \right] (2D)$$
PROBLEM I

Instantaneous and Finite-Time Bounds for Growth of Enstrophy in 1D Burgers Problem
B. Protas & D. Ayala

Sharpness of Estimates as Optimization Problem
Bounds for 2D Navier-Stokes Problem
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Finite-Time Bounds in 1D Burgers Problem

Burgers equation \((\Omega = [0, 1], \ u : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R})\)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } \Omega
\]

\[
u \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{at } \ t = 0
\]

Periodic B.C.

- Solutions smooth for all times
- Questions of sharpness of enstrophy estimates still relevant

\[
\frac{d \mathcal{E}(t)}{dt} \leq \frac{3}{2} \left( \frac{1}{\pi^2 \nu} \right)^{1/3} \mathcal{E}(t)^{5/3}
\]

- Best available finite-time estimate

\[
\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[ \mathcal{E}_0^{1/3} + \left( \frac{L}{4} \right)^2 \left( \frac{1}{\pi^2 \nu} \right)^{4/3} \mathcal{E}_0 \right]^3 \quad \xrightarrow{\mathcal{E}_0 \to \infty} C_2 \mathcal{E}_0^3
\]
Burgers equation \((\Omega = [0, 1], \ u : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R})\)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } \Omega \\

u(x) = \phi(x) \quad \text{at } t = 0
\]

Periodic B.C.

Enstrophy : \(\mathcal{E}(t) = \frac{1}{2} \int_0^1 |u_x(x, t)|^2 \, dx\)

- Solutions smooth for all times
- Questions of sharpness of enstrophy estimates still relevant

\[
\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left( \frac{1}{\pi^2 \nu} \right)^{1/3} \mathcal{E}(t)^{5/3}
\]

- Best available finite-time estimate

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\]

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B. Protas & D. Ayala

Extreme Vortices & the Blow-Up Problem

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Finite-time Estimates — a different approach without explicit time-integration of instantaneous estimates

Spectral Properties of Solutions of Burgers Equation with Small Dissipation

Andrei Birăăuc

1 Introduction

This present paper concerns the initial value problem for the one dimensional (dim \( x = 1 \)) parabolic equation of Burgers type:

\[
\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = \delta u_{xx} ,
\]

with the initial state

\[
u(0, x) = u_0(x) ,
\]

Still unclear whether the resulting finite-time estimate is (much) sharper ...
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Andrei Biryuk

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"Small" Problem of Lu & Doering (2008), I

- Estimate $\frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^{5/3}$ at a fixed instant of time $t$

$$\max_{u \in H^1(\Omega)} \frac{d\mathcal{E}(t)}{dt}$$

subject to $\mathcal{E}(t) = \mathcal{E}_0$

where

$$\frac{d\mathcal{E}(t)}{dt} = -\nu \left\| \frac{\partial^2 u}{\partial x^2} \right\|_2^2 + \frac{1}{2} \int_0^1 \left( \frac{\partial u}{\partial x} \right)^3 d\Omega$$

- $\mathcal{E}_0$ is a parameter

- Solution (maximizing field) found analytically!
  (in terms of elliptic integrals and Jacobi elliptic functions)
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\text{subject to } \mathcal{E}(t) = \mathcal{E}_0
\]

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\]

\( \mathcal{E}_0 \) is a parameter

Solution (maximizing field) found analytically!
(in terms of elliptic integrals and Jacobi elliptic functions)
"Small" Problem of Lu & Doering (2008), II

\[
\left[ \frac{d \mathcal{E}(t)}{dt} \right]_{max} = 0.2476 \mathcal{E}_0^{5/3} \nu^{1/3}
\]

Instantaneous estimate is sharp
\[ \left[ \frac{d\mathcal{E}(t)}{dt} \right]_{\text{max}} = 0.2476 \frac{\mathcal{E}_0^{5/3}}{\nu^{1/3}} \]

instantaneous estimate is sharp

\[ \max_{t \geq 0} E(t) \sim C \mathcal{E}_0^{1.048} \]

finite-time estimate
(far from saturated)
Finite-Time Optimization Problem (I)

▶ Statement

$$\max_{\phi \in H^1(\Omega)} \mathcal{E}(T)$$

subject to \( \mathcal{E}(t) = \mathcal{E}_0 \)

\( T, \mathcal{E}_0 \) — parameters

▶ Optimality Condition

$$\forall \phi' \in H^1 \quad J'_\lambda(\phi; \phi') = - \int_0^1 \frac{\partial^2 u}{\partial x^2} \bigg|_{t=T} u' \bigg|_{t=T} \, dx - \lambda \int_0^1 \frac{\partial^2 \phi}{\partial x^2} \bigg|_{t=0} u' \bigg|_{t=0} \, dx$$

\( \lambda \) — Lagrange multiplier
Finite-Time Optimization Problem (I)

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\[
\max_{\phi \in H^1(\Omega)} \mathcal{E}(T)
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\[
\text{subject to } \mathcal{E}(t) = \mathcal{E}_0
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\]

\[\lambda \text{ — Lagrange multiplier}\]
Finite-Time Optimization Problem (II)

- **Gradient Descent**

\[ \phi^{(n+1)} = \phi^{(n)} - \tau^{(n)} \nabla J(\phi^{(n)}), \quad n = 1, \ldots, \]
\[ \phi^{(0)} = \phi_0, \]

- **Step size** \( \tau^{(n)} \) found via *arc minimization*
Finite-Time Optimization Problem (II)

► Gradient Descent

\[ \phi^{(n+1)} = \phi^{(n)} - \tau^{(n)} \nabla J(\phi^{(n)}), \quad n = 1, \ldots, \]

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where \( \nabla J \) determined from adjoint system via \( H^1 \) Sobolev preconditioning

\[ - \frac{\partial u^*}{\partial t} - u \frac{\partial u^*}{\partial x} - \nu \frac{\partial^2 u^*}{\partial x^2} = 0 \quad \text{in } \Omega \]

\[ u^*(x) = -\frac{\partial^2 u}{\partial x^2}(x) \text{ at } t = T \]

Periodic B.C.

► Step size \( \tau^{(n)} \) found via arc minimization
Finite-Time Optimization Problem (II)

- **Gradient Descent**
  \[
  \phi^{(n+1)} = \phi^{(n)} - \tau^{(n)} \nabla J(\phi^{(n)}), \quad n = 1, \ldots,
  \]
  \[
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  \]
  where \( \nabla J \) determined from *adjoint system* via \( H^1 \) Sobolev preconditioning

\[
- \frac{\partial u^*}{\partial t} - u \frac{\partial u^*}{\partial x} - \nu \frac{\partial^2 u^*}{\partial x^2} = 0 \quad \text{in } \Omega
\]

\[
u = \frac{\partial^2 u}{\partial x^2}(x) \quad \text{at } t = T
\]

- **Step size** \( \tau^{(n)} \) found via *arc minimization*

Two parameters: $T, E_0$ ($\nu = 10^{-3}$)

Optimal initial conditions corresponding to initial guess with wavenumber $m = 1$ (local maximizers)
Two parameters: $T, \mathcal{E}_0$ ($\nu = 10^{-3}$)

Optimal initial conditions corresponding to initial guess with wavenumber $m = 1$ (local maximizers)

Fixed $\mathcal{E}_0 = 10^3$, different $T$
Two parameters: $T, \mathcal{E}_0 \quad (\nu = 10^{-3})$

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Fixed $\mathcal{E}_0 = 10^3$, different $T$

Fixed $T = 0.0316$, different $\mathcal{E}_0$
Sharpness of Estimates as Optimization Problem
Bounds for 2D Navier-Stokes Problem
Bounds for 3D Navier-Stokes Problem

Finite-Time Bounds in 1D Burgers Problem

Regularity Problem for Navier-Stokes Equation
Research Program and Earlier Results

B. Protas & D. Ayala
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$\max_{t \in [0, T]} E(t)$ versus $T$

$E(T)$ versus $E_0$

$E_\max$ versus $E_0$

$\max_{t \in [0, T]} E(t)$ versus $E_0$
Extreme Vortices & the Blow-Up Problem

\[ \max_{t \in [0, T]} \mathcal{E}(t) \sim C \mathcal{E}_0^{1.5} \]

\[ \max_{t \in [0, T]} \mathcal{E}(t) \text{ versus } \mathcal{E}_0 \]

\[ \max_{t \in [0, T]} \mathcal{E}(t) \text{ versus } T \]

\[ \mathcal{E}(T^*) \text{ versus } \mathcal{E}_0 \]
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\[ \max_{t \in [0, T]} \mathcal{E}(t) \text{ versus } T \]

\[ \mathcal{E}(T) \text{ versus } \mathcal{E}_0 \]

\[ \arg \max_{t \in [0, T]} \mathcal{E}(t) \text{ versus } \mathcal{E}_0 \]

\[ \max_{t \in [0, T]} \mathcal{E}(t) \sim C \mathcal{E}_0^{-0.5} \]

argmax_{t \in [0, T]} \mathcal{E}(t) \sim C \mathcal{E}_0^{1.5}

B. Protas & D. Ayala
Extreme Vortices & the Blow-Up Problem
Sol’ns found with initial guesses $\phi^{(m)}(x) = \sin(2\pi mx), \ m = 1, 2, \ldots$

$m = 1, \mathcal{E}_0 = 10^3$

$m = 2, \mathcal{E}_0 = 10^3$

Change of variables leaving Burgers equation invariant ($L \in \mathbb{Z}^+$):

$x = L\xi, \ (x \in [0,1], \ \xi \in [0,1/L]), \quad \tau = t/L^2$

$v(\tau, \xi) = Lu(x(\xi), t(\tau)), \quad \mathcal{E}_v(\tau) = L^4\mathcal{E}_u\left(\frac{t}{L^2}\right)$
Sol'ns found with initial guesses $\phi^{(m)}(x) = \sin(2\pi mx)$, $m = 1, 2, \ldots$

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Solutions for $m = 1$ and $m = 2$, after rescaling

Using initial guess: $\phi^{(0)}(x) = \sin(2\pi mx)$, $m = 1$, or $m = 2$

All local maximizers with $m = 2, 3, \ldots$ are rescaled copies of the $m = 1$ maximizer.
- Solutions for $m = 1$ and $m = 2$, after rescaling

- Using initial guess:
  \[ \phi(0)(x) = \epsilon \sin(2\pi mx) + (1 - \epsilon) \sin(2\pi nx), \quad m \neq n, \quad \epsilon > 0 \]

- All local maximizers with $m = 2, 3, \ldots$ are rescaled copies of the $m = 1$ maximizer
Solutions for $m = 1$ and $m = 2$, after rescaling

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## Relevant Estimates

<table>
<thead>
<tr>
<th>Equation</th>
<th>Best Estimate</th>
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</tr>
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<tbody>
<tr>
<td><strong>1D Burgers</strong></td>
<td>$\frac{dE(t)}{dt} \leq \frac{3}{2} \left( \frac{1}{\pi^2 \nu} \right)^{1/3} E(t)^{5/3}$</td>
<td><strong>YES</strong> Lu &amp; Doering (2008)</td>
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<td><strong>1D Burgers</strong></td>
<td>$\max_{t \in [0, T]} E(t) \leq \left[ E_0^{1/3} + \left( \frac{L}{4} \right)^2 \left( \frac{1}{\pi^2 \nu} \right)^{4/3} E_0 \right]^3$</td>
<td><strong>NO</strong> Ayala &amp; P. (2011)</td>
</tr>
<tr>
<td><strong>2D Navier–Stokes</strong></td>
<td>$\frac{dP(t)}{dt} \leq - \left( \frac{\nu}{E} \right) P^2 + C_1 \left( \frac{E}{\nu} \right) P$</td>
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<td>$\max_{t &gt; 0} P(t) \leq \left[ P_0^{1/2} + \frac{C_2}{4\nu^2} K_0^{1/2} E_0 \right]^2$</td>
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<td>$\frac{dE(t)}{dt} \leq \frac{27C^2}{128\nu^3} E(t)^3$</td>
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<td>$E(t) \leq \frac{E(0)}{\sqrt{1 - 4 \frac{C E(0)^2}{\nu^3} t}}$</td>
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**B. Protas & D. Ayala** Extreme Vortices & the Blow-Up Problem

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**Sharpness of Estimates as Optimization Problem**

**Bounds for 2D Navier–Stokes Problem**

**Bounds for 3D Navier–Stokes Problem**

**Regularity Problem for Navier–Stokes Equation**

**Research Program and Earlier Results**

**Finite-Time Bounds in 1D Burgers Problem**
PROBLEM II

Instantaneous Bounds for Growth of Palinstrophy in 2D Navier-Stokes Problem
# Relevant Estimates

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<td>Yes Lu &amp; Doering (2008)</td>
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<td>instantaneous</td>
<td>$\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[ \mathcal{E}_0^{1/3} + \left( \frac{L}{4} \right)^2 \left( \frac{1}{\pi^2 \nu} \right)^{4/3} \mathcal{E}_0 \right]^3$</td>
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<td>$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1-4C\mathcal{E}(0)^2/\nu^3} - t}$</td>
<td></td>
</tr>
</tbody>
</table>
2D vorticity equation in a periodic box \((\omega = e_z \cdot \omega)\)

\[
\frac{\partial \omega}{\partial t} + J(\omega, \psi) = \nu \Delta \omega \quad \text{where} \quad J(f, g) = f_x g_y - f_y g_x
\]

\[-\Delta \psi = \omega\]

Enstrophy uninteresting in 2D flows (w/o boundaries)

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \omega^2 \, d\Omega = -\nu \int_{\Omega} (\nabla \omega)^2 \, d\Omega < 0
\]

Evolution equation for the vorticity gradient \(\nabla \omega\)

\[
\frac{\partial \nabla \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \nabla \omega = \nu \Delta \nabla \omega + \nabla \omega \cdot \nabla \mathbf{u}
\]

"stretching" term
2D vorticity equation in a periodic box \((\omega = e_z \cdot \omega)\)

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\]
Palinstrophy

\[ \mathcal{P}(t) \triangleq \int_{\Omega} (\nabla \omega(t, x))^2 \, d\Omega = \int_{\Omega} (\nabla \Delta \psi(t, x))^2 \, d\Omega \]

Also of interest — Kinetic Energy

\[ \mathcal{K}(t) \triangleq \int_{\Omega} u(t, x)^2 \, d\Omega = \int_{\Omega} (\nabla \psi(t, x))^2 \, d\Omega \]

Poincaré’s inequality

\[ \mathcal{K} \leq (2\pi)^{-2} \mathcal{E} \leq (2\pi)^{-2} \mathcal{P} \]
Palinstrophy

$\mathcal{P}(t) \triangleq \int_{\Omega} (\nabla \omega(t, \mathbf{x}))^2 \, d\Omega = \int_{\Omega} (\nabla \Delta \psi(t, \mathbf{x}))^2 \, d\Omega$

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$\mathcal{K}(t) \triangleq \int_{\Omega} u(t, \mathbf{x})^2 \, d\Omega = \int_{\Omega} (\nabla \psi(t, \mathbf{x}))^2 \, d\Omega$

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\[ \mathcal{K} \leq (2\pi)^{-2} \mathcal{E} \leq (2\pi)^{-2} \mathcal{P} \]
Estimates for the Rate of Growth of Palinstrophy

\[
\frac{d\mathcal{P}(t)}{dt} = \int_{\Omega} J(\Delta \psi, \psi) \Delta^2 \psi \, d\Omega - \nu \int_{\Omega} (\Delta^2 \psi)^2 \, d\Omega \triangleq \mathcal{R}_\mathcal{P}(\psi)
\]

Using Poincaré’s inequality (may not be sharp)

\[
\frac{d\mathcal{P}(t)}{dt} \leq \frac{C}{\nu} \mathcal{P}^2,
\]

Bound on growth in finite time

\[
\max_{t>0} \mathcal{P}(t) \leq \left[ \mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2} \mathcal{K}_0^{1/2} \mathcal{E}_0 \right]^2 \quad \text{(Ayala, 2012)}
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\]

\[
\frac{d\mathcal{P}(t)}{dt} \leq -\left( \frac{\nu}{\mathcal{E}} \right) \mathcal{P}^2 + C_1 \left( \frac{\mathcal{E}}{\nu} \right) \mathcal{P}
\]

(Doering & Lunasin, 2011)

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\[ \frac{d\mathcal{P}(t)}{dt} \leq -\left( \frac{\nu}{\mathcal{E}} \right) \mathcal{P}^2 + C_1 \left( \frac{\mathcal{E}}{\nu} \right) \mathcal{P} \quad \text{(Doering & Lunasin, 2011)} \]

\[ \frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu} K^{1/2} \mathcal{P}^{3/2} \quad \text{(Ayala, 2012)} \]

Using Poincaré’s inequality (may not be sharp)

\[ \frac{d\mathcal{P}(t)}{dt} \leq \frac{C}{\nu} \mathcal{P}^2, \]

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\[ \max_{t>0} \mathcal{P}(t) \leq \left[ \mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2} K_0^{1/2} \mathcal{E}_0 \right]^2 \quad \text{(Ayala, 2012)} \]
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\]

\[
\frac{d\mathcal{P}(t)}{dt} \leq - \left( \frac{\nu}{\mathcal{E}} \right) \mathcal{P}^2 + C_1 \left( \frac{\mathcal{E}}{\nu} \right) \mathcal{P}
\]  
(Doering & Lunasin, 2011)

\[
\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu} \mathcal{K}^{1/2} \mathcal{P}^{3/2}
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Maximum Growth of $\frac{dP(t)}{dt}$ for fixed $\mathcal{E}_0 > 0, P_0 > (2\pi)^2 \mathcal{E}_0$

$$\max_{\psi \in \mathcal{S}_{P_0, \mathcal{E}_0}} \mathcal{R}_{P_0}(\psi)$$

where

$$\mathcal{S}_{P_0, \mathcal{E}_0} = \left\{ \psi \in H^4(\Omega) : \begin{cases} \frac{1}{2} \int_{\Omega} (\nabla \Delta \psi)^2 \, d\Omega = P_0 \\ \frac{1}{2} \int_{\Omega} (\Delta \psi)^2 \, d\Omega = \mathcal{E}_0 \end{cases} \right\}$$

Maximum Growth of $\frac{dP(t)}{dt}$ for fixed $\mathcal{K}_0 > 0, P_0 > (2\pi)^4 \mathcal{K}_0$

$$\max_{\psi \in \mathcal{S}_{P_0, \mathcal{K}_0}} \mathcal{R}_{P_0}(\psi)$$

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\frac{1}{2} \int_{\Omega} (\nabla \Delta \psi)^2 \, d\Omega = \mathcal{P}_0 \\
\frac{1}{2} \int_{\Omega} (\Delta \psi)^2 \, d\Omega = \mathcal{E}_0
\end{array} \right\}
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\end{array} \right\}
\]
Small Palinstrophy Limit: \( P_0 \to (2\pi)^2 \mathcal{E}_0 \)

\[
\tilde{\varphi}_0 = \arg \max_{\varphi \in S_0} \mathcal{R}_0(\varphi), \quad \mathcal{R}_0(\varphi) = -\nu \int_\Omega (\Delta^2 \varphi)^2 \, d\Omega,
\]

\[
S_0 = \left\{ \varphi \in H^4(\Omega) : \frac{1}{2} \int_\Omega (\nabla \Delta \psi)^2 \, d\Omega = \frac{(2\pi)^2}{2} \int_\Omega (\Delta \psi)^2 \, d\Omega \right\}
\]

Optimizers: Eigenfunctions of the Laplacian \( (\tilde{\varphi}_0 \in \text{Ker}(\Delta)) \)
**Small Palinstrophy Limit:**

\[ \mathcal{P}_0 \rightarrow (2\pi)^2 \mathcal{E}_0 \]

\[ \tilde{\varphi}_0 = \arg \max_{\varphi \in S_0} R_0(\varphi), \quad R_0(\varphi) = -\nu \int_{\Omega} (\Delta^2 \varphi)^2 \, d\Omega, \]

\[ S_0 = \left\{ \varphi \in H^4(\Omega) : \frac{1}{2} \int_{\Omega} (\nabla \Delta \psi)^2 \, d\Omega = \frac{(2\pi)^2}{2} \int_{\Omega} (\Delta \psi)^2 \, d\Omega \right\} \]

**Optimizers:**

\[ \text{Eigenfunctions of the Laplacian} \quad (\tilde{\varphi}_0 \in \text{Ker}(\Delta)) \]

\[ \varphi(x, y) = \sin(\pi(y - x)) \sin(\pi(y + x)) \]

\[ \varphi(x, y) = \sin(2\pi x) \sin(2\pi y) \]
Numerical Solution of Maximization Problem

- Discretization of Gradient Flow

\[ \frac{d\psi}{d\tau} = -\nabla^{H^4} R_{\nu}(\psi), \quad \psi(0) = \psi_0 \]

- Gradient in $H^4(\Omega)$ (via variational techniques)

\[
\begin{bmatrix}
\text{Id} - L^8 \Delta^4 \\
\end{bmatrix}
\nabla^{H^4} R_{\nu} = \nabla^{L^2} R_{\nu} \quad \text{(Periodic BCs)}
\]

\[
\nabla^{L^2} R_{\nu}(\psi) = \Delta^2 J(\Delta \psi, \psi) + \Delta J(\psi, \Delta^2 \psi) + J(\Delta^2 \psi, \Delta \psi) - 2\nu \Delta^4 \psi
\]

- Constraint satisfaction via arc minimization
Numerical Solution of Maximization Problem

- Discretization of Gradient Flow

\[ \frac{d\psi}{d\tau} = -\nabla^{H^4} R_\nu(\psi), \quad \psi(0) = \psi_0 \]
\[ \psi^{(n+1)} = \psi^{(n)} - \Delta \tau^{(n)} \nabla^{H^4} R_\nu(\psi^{(n)}), \quad \psi^{(0)} = \psi_0 \]

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Numerical Solution of Maximization Problem

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\]

- Constraint satisfaction via arc minimization
Maximizers with Fixed \((K_0, P_0)\)

Estimate:
\[
\frac{dP(t)}{dt} \leq \frac{C_2}{\nu} K_0^{1/2} P_0^{3/2}
\]

\[
\max \frac{dP}{dt} \text{ vs. } P_0, K_0 = 10
\]

(b) \(P_0 \approx 10P_c\)

(c) \(P_0 \approx 10^4P_c\)
Maximizers with Fixed \((K_0, P_0)\)

Estimate: \[
\frac{dP(t)}{dt} \leq \frac{C_2}{\nu} K_0^{1/2} P_0^{3/2}
\]

\[
\max \frac{dP}{dt} \text{ vs. } P_0, K_0 = 10
\]

(a) \(P_0 \approx 10P_c\)  
(b) \(P_0 \approx 10P_c\)

(c) \(P_0 \approx 10^4P_c\)  
(d) \(P_0 \approx 10^4P_c\)
Maximizers with Fixed \((K_0, P_0)\)

Estimate: \[
\frac{dP(t)}{dt} \leq \frac{C_2}{\nu} K_0^{1/2} P_0^{3/2}
\]

\[
\max \frac{dP}{dt} \text{ vs. } P_0, K_0 = 10
\]

\[
\max \frac{dP}{dt} \sim P_0^{3/2} \text{ as } P_0 \to \infty
\]

\(\text{ (a) } P_0 \approx 10P_c\)

\(\text{ (b) } P_0 \approx 10P_c\)

\(\text{ (c) } P_0 \approx 10^4P_c\)

\(\text{ (d) } P_0 \approx 10^4P_c\)
Maximizers with Fixed \((\mathcal{E}_0, \mathcal{P}_0)\)

Estimate: \[
\frac{d\mathcal{P}(t)}{dt} \leq -\left(\frac{\nu}{\mathcal{E}_0}\right) \mathcal{P}_0^2 + C_1 \left(\frac{\mathcal{E}_0}{\nu}\right) \mathcal{P}_0
\]

\[
\max \frac{d\mathcal{P}}{dt} \text{ vs. } \mathcal{P}_0, \mathcal{E}_0 = 10^3
\]

(a) \(\mathcal{P}_0 \approx \mathcal{P}_c\)

(b) \(\mathcal{P}_0 \approx 10\mathcal{P}_c\)

(c) \(\mathcal{P}_0 \approx 10^2\mathcal{P}_c\)

(d) \(\mathcal{P}_0 \approx 10^{7/2}\mathcal{P}_c\)
Maximizers with Fixed \((E_0, P_0)\)

Estimate: \[
\frac{dP(t)}{dt} \leq -\left(\frac{\nu}{E_0}\right)P_0^2 + C_1\left(\frac{E_0}{\nu}\right)P_0
\]

\[
\max \frac{dP}{dt} \text{ vs. } P_0, \ E_0 = 10^3
\]

\[
\max \frac{dP}{dt} \text{ vs. } P_0
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B. Protas & D. Ayala

Extreme Vortices & the Blow-Up Problem
Maximizers with Fixed \((K_0, P_0)\)

Finite-Time Estimate: \(\max_{t > 0} P(t) \leq \left[ P_0^{1/2} + \frac{C_2}{4^{1/2}} K_0^{1/2} E_0 \right]^2\)

---

\(P_0\)-constraint

---

\(\{K_0, P_0\}\)-constraint

(a) \(t = 0.000213\)

(b) \(t = 0.000458\)

(c) \(t = 0.000633\)

(d) \(t = 0.001265\)
## Relevant Estimates

<table>
<thead>
<tr>
<th></th>
<th>Best Estimate</th>
<th>Sharp?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1D Burgers</strong></td>
<td>$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left( \frac{1}{\pi^2 \nu} \right)^{1/3} \mathcal{E}(t)^{5/3}$</td>
<td>Yes Lu &amp; Doering (2008)</td>
</tr>
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<td>instantaneous</td>
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</tr>
<tr>
<td><strong>1D Burgers</strong></td>
<td>$\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[ \mathcal{E}_0^{1/3} + \left( \frac{L}{4} \right)^2 \left( \frac{1}{\pi^2 \nu} \right)^{4/3} \mathcal{E}_0 \right]^3$</td>
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<td><strong>2D Navier–Stokes</strong></td>
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<td>[Yes] Ayala &amp; P. (2013)</td>
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<td>Yes Lu &amp; Doering (2008)</td>
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<tr>
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<tr>
<td></td>
<td>$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4 \frac{C \mathcal{E}(0)^2}{\nu^3} t}}$</td>
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## Relevant Estimates

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<tr>
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<td>( \frac{dE(t)}{dt} \leq \frac{3}{2} \left( \frac{1}{\pi^2 \nu} \right)^{1/3} E(t)^{5/3} )</td>
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<td>finite–time</td>
<td>[ \max_{t \in [0, T]} E(t) \leq \left[ E_0^{1/3} + \left( \frac{L}{4} \right)^2 \left( \frac{1}{\pi^2 \nu} \right)^{4/3} E_0 \right]^3 ]</td>
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<td>( \frac{dE(t)}{dt} \leq \frac{27 C^2}{128 \nu^3} E(t)^3 )</td>
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<td>finite–time</td>
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</tbody>
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**B. Protas & D. Ayala**

**Extreme Vortices & the Blow-Up Problem**
PROBLEM III

Instantaneous Bounds for Growth of Enstrophy in 3D Navier-Stokes Problem

(Preliminary Results)
## Relevant Estimates

<table>
<thead>
<tr>
<th></th>
<th><strong>Best Estimate</strong></th>
<th><strong>Sharp?</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1D Burgers</strong></td>
<td>$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left( \frac{1}{\pi^2\nu} \right)^{1/3} \mathcal{E}(t)^{5/3}$</td>
<td><strong>YES</strong> Lu &amp; Doering (2008)</td>
</tr>
<tr>
<td><strong>finite–time</strong></td>
<td>$\max_{t \in [0,T]} \mathcal{E}(t) \leq \left[ \mathcal{E}_0^{1/3} + \left( \frac{L}{4} \right)^2 \left( \frac{1}{\pi^2\nu} \right)^{4/3} \mathcal{E}_0 \right]^3$</td>
<td><strong>NO</strong> Ayala &amp; P. (2011)</td>
</tr>
<tr>
<td><strong>2D Navier–Stokes</strong></td>
<td>$\frac{d\mathcal{P}(t)}{dt} \leq -\left( \frac{\nu}{\mathcal{E}} \right) \mathcal{P}^2 + C_1 \left( \frac{\mathcal{E}}{\nu} \right) \mathcal{P}$</td>
<td>[<strong>YES</strong>] Ayala &amp; P. (2013)</td>
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<td><strong>instantaneous</strong></td>
<td>$\frac{d\mathcal{P}(t)}{dt} \leq \frac{C_2}{\nu} \mathcal{K}^{1/2} \mathcal{P}^{3/2}$</td>
<td>[<strong>YES</strong>] Ayala &amp; P. (2013)</td>
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<td><strong>finite–time</strong></td>
<td>$\max_{t &gt; 0} \mathcal{P}(t) \leq \left[ \mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2} \mathcal{K}_0^{1/2} \mathcal{E}_0 \right]^2$</td>
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**B. Protas & D. Ayala**

**Extreme Vortices & the Blow-Up Problem**
Rate of Growth of Enstrophy

\[
\frac{d\mathcal{E}}{dt} = -\nu \int_{\Omega} |\Delta u|^2 \, dx + \int_{\Omega} u \cdot \nabla u \cdot \Delta u \, dx \triangleq \mathcal{R}_{\mathcal{E}_0}(u)
\]

Best available instantaneous upper bound

\[
\frac{d\mathcal{E}}{dt} \leq \frac{C}{\nu^3} \mathcal{E}^3
\]

Finite-time estimates
Rate of Growth of Enstrophy

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Finite-time estimates
Rate of Growth of Enstrophy

\[ \frac{d\mathcal{E}}{dt} = -\nu \int_{\Omega} |\Delta u|^2 \, dx + \int_{\Omega} u \cdot \nabla u \cdot \Delta u \, dx \triangleq R_{\mathcal{E}_0}(u) \]

Best available instantaneous upper bound

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Finite-time estimates

\[ \max_{t \geq 0} \mathcal{E}(t) \leq \frac{\mathcal{E}_0}{\sqrt{1 - \frac{4C_3}{\nu^3} \mathcal{E}_0^2 t}} \]
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\[ \max_{t \geq 0} \mathcal{E}(t) \leq \frac{\mathcal{E}_0}{\sqrt{1 - \frac{4C_3}{\nu^3} \mathcal{E}_0^2 t}} \]

\[ \frac{1}{\mathcal{E}(0)} - \frac{1}{\mathcal{E}(t)} \leq \frac{27}{(2\pi\nu)^4} [\mathcal{K}(0) - \mathcal{K}(t)] \]
Single Constraint: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$

\[
\max_{u \in S_{\mathcal{E}_0}} \mathcal{R}_{\mathcal{E}_0}(u) \quad \text{where}
\]

\[
S_{\mathcal{E}_0} = \{ u \in H^2(\Omega) : \nabla \cdot u = 0, \mathcal{E}(u) = \mathcal{E}_0 \}
\]

Two Constraints: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$ and $\mathcal{K}_0 < (2\pi)^{-2}\mathcal{E}_0$

\[
\max_{u \in S_{\mathcal{K}_0, \mathcal{E}_0}} \mathcal{R}_{\mathcal{E}_0}(u) \quad \text{where}
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\[
S_{\mathcal{K}_0, \mathcal{E}_0} = \{ u \in H^2(\Omega) : \nabla \cdot u = 0, \mathcal{K}(u) = \mathcal{K}_0, \mathcal{E}(u) = \mathcal{E}_0 \}
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Numerical solution via discretized gradient flow
(requires resolutions up to $512^3$)
▶ Single Constraint: maximum rate of growth $\frac{d\mathcal{E}(t)}{dt}$ for fixed $\mathcal{E}_0 > 0$

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Extreme Vortex States for $\mathcal{E}_0 \to 0$ (single constraint)

- In the limit $\mathcal{E}_0 \to 0$ optimal states found analytically
  $\implies$ div-free eigenfunctions of vector Laplacian (3 branches)

- Case (a): Largest value of $d\mathcal{E}/dt$
- Case (c): Taylor-Green vortex (Taylor & Green 1937)
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- In the limit $\mathcal{E}_0 \to 0$ optimal states found analytically
  \[ \Rightarrow \text{div-free eigenfunctions of vector Laplacian (3 branches)} \]

(a) $|\mathbf{k}|^2 = 1$

(b) $|\mathbf{k}|^2 = 2$

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Single-constraint maximizers $\tilde{u}_{E_0}$ (Lu & Doering 2008)

(a) $E_0 = 1 \times 10^{-2}$
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(d) $\mathcal{E}_0 = 20$
(e) $\mathcal{E}_0 = 50$
(f) $\mathcal{E}_0 = 100$
Two-constraint maximizers $\mathbf{u}_{K_0, E_0}$ $(K_0 = 1)$

**Graph:**
- $E_0 \approx 40$
- $E_0 \approx 90$
- $E_0 \approx 200$
Time evolution of $\tilde{u}_{\mathcal{E}_0}$ (single constraint: $\mathcal{E}_0 = 100$)

(a) $t = 0.0$  (b) $t = 1.1 \times 10^{-4}$  (c) $t = 1.75 \times 10^{-3}$

(d) $t = 8.63 \times 10^{-3}$  (e) $t = 5.74 \times 10^{-2}$  (f) $t = 0.198$
(a) single constraint ($\mathcal{E}_0 = 60$)

- extreme (instantaneously optimal) states $\tilde{\mathbf{u}}_{\mathcal{E}_0}$,
- - - Taylor-Green vortex
- - - Kida-Pelz vortex
(a) single constraint ($E_0 = 60$)

(b) two constraints ($K_0 = 1, E_0 = 64$)

- extreme (instantaneously optimal) states $\tilde{u}_{E_0}, \tilde{u}_{K_0,E_0}$
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\[
\frac{1}{\mathcal{E}(0)} - \frac{1}{\mathcal{E}(t)} \leq C \left[ \mathcal{K}(0) - \mathcal{K}(t) \right]
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\]

(a) single constraint \((\mathcal{E}_0 = 60)\)  
(b) two constraints \((\mathcal{K}_0 = 1, \mathcal{E}_0 = 64)\)
\[
\frac{d\mathcal{E}}{dt} \leq C' \mathcal{E}^3 \quad \Rightarrow \quad \frac{1}{\mathcal{E}(0)} - \frac{1}{\mathcal{E}(t)} \leq C \left[ \mathcal{K}(0) - \mathcal{K}(t) \right]
\]

\[
\Rightarrow \quad \max_{t \geq 0} \mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{1 - C \mathcal{K}(0) \mathcal{E}(0)}, \quad C, C' - \text{numerical fit}
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Conclusions

- Found extreme vortex states in 2D and 3D saturating the worst-case mathematical bounds (although, in contrast to some recent studies in 3D, e.g., Bustamante & Brachet 2012, Kerr et al. 2013, 2014, the Reynolds numbers are small).

- Identified regions in the initial data phase-space \( \{K_0, E_0\} \) for which global regularity is guaranteed.

- So far, no evidence of blow-up in 3D, although due to small Reynolds numbers, the results are not conclusive.

- Need to solve the optimization problem on finite time windows, i.e.,

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\max_{u_0} \mathcal{E}(T)
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- Why do two-constraint optimizers exhibit a larger finite-time growth in 2D, but not in 3D? (finite Re effect?)

- Regularity problem for 3D Euler equation.

- Singularity formation in “active scalar” equations (fractional Burgers equation, surface quasi-geostrophic equation, etc.).

- Extreme behavior in the presence of noise.

- Extreme states with more complex structure: simultaneously maximize $\mathcal{R}_{\epsilon_0}(u)$ and helicity $\mathcal{H}(u)$ (via multiobjective optimization)
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