Remarks on the Exponential Rules in Linear Logic

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We work in linear logic with weakening and consider the standard exponential rules

\[\begin{align*}
\Gamma, !A, !A \Rightarrow \Delta & \quad \text{(lc)} \\
\Gamma, A \Rightarrow \Delta & \quad \text{(ld)} \\
\Gamma, !A \Rightarrow \Delta & \quad \text{(!R)}
\end{align*}\]

The rules (lc) and (ld) say that contraction is applicable to formulas prefixed with ‘bang’. In the intuitive interpretation of formulas as resources, !A thus denotes a resource which can be used arbitrarily often.

In the same spirit, the soundness of the rule (!R) is sometimes argued for as follows: If we can obtain the resource A from resources !Γ, then we can repeat this ‘process’ to obtain arbitrarily many A’s, since we never run out of the unbounded resources in !Γ.

Since the resource interpretation of linear logic is not formalized, no direct argument for the completeness of the rule (!R) can be made. In fact, some derivations seem to conflict with the interpretation of !A as ‘arbitrarily many’.

For example, we can prove

\[C, !(C \rightarrow A \otimes C) \Rightarrow \underbrace{A \otimes \ldots \otimes A}_n\]

for every \(n\), but we cannot prove

\[C, !(C \rightarrow A \otimes C) \Rightarrow !A\]

It is thus not clear which notion of ‘arbitrarily many’ is captured by the rule (!R).

We try to gain some insight into this problem by comparing (!R) to a naive infinitary rule of the form

\[\begin{align*}
\Gamma \Rightarrow A & \quad \Gamma \Rightarrow A \otimes A \\
\Gamma \Rightarrow A \otimes A & \quad \Gamma \Rightarrow A \otimes A \otimes A \\
\ldots & \\
\Gamma \Rightarrow !A
\end{align*}\]