Nonlinear dispersive asymptotic models for the propagation of internal waves

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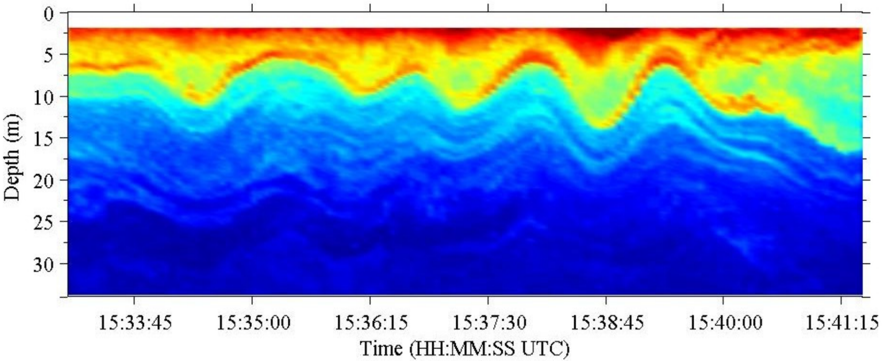
Internal gravity waves

Stratification, due to variation of salinity and temperature. [1]



1. Credits : St. Lawrence Estuary Internal Wave Experiment (SLEIWEX) http://myweb.dal.ca/kelley/SLEIWEX/index.php

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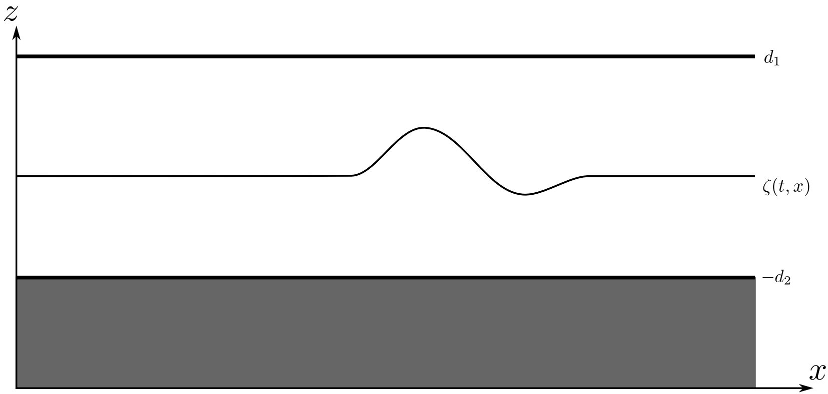
The full Euler system



- Horizontal dimension $d = 1$, flat bottom, rigid lid.
- Irrotational, incompressible, inviscid, immiscible fluids.
- Fluids at rest at infinity, (very small) surface tension.

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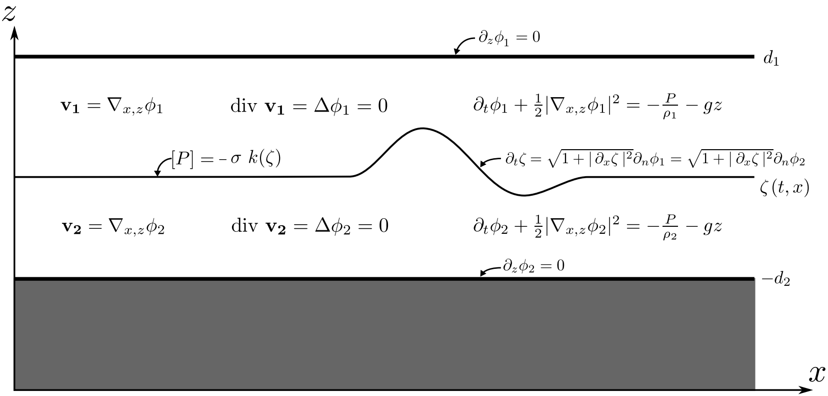

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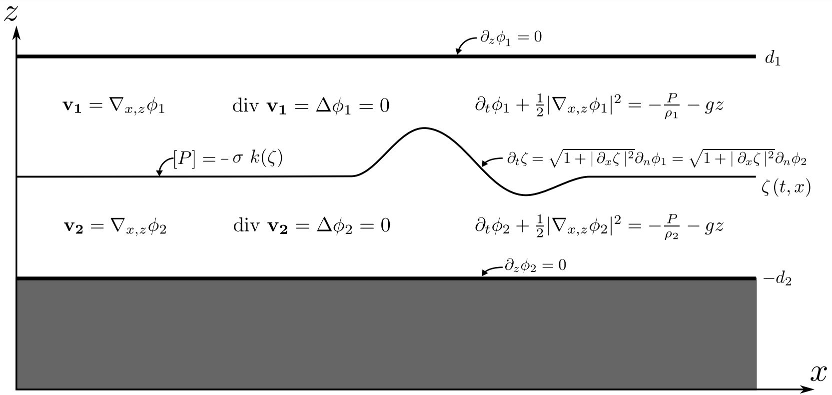


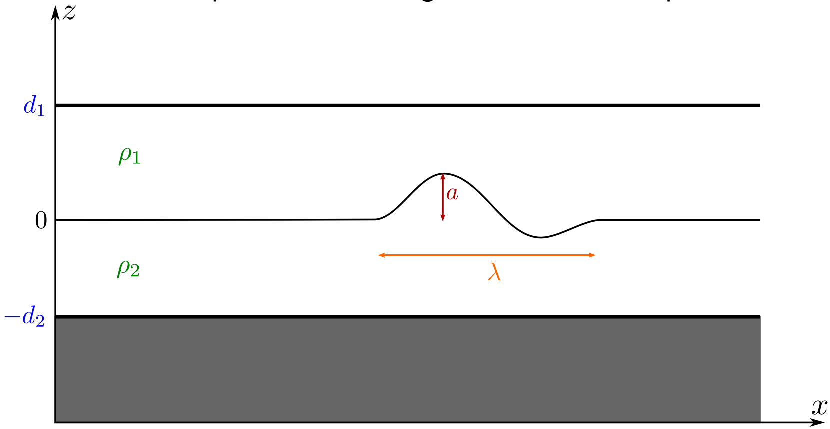
The system can be rewritten as two coupled evolution equations in

$$\zeta \quad \text{and} \quad \psi \equiv \phi_{2|\text{interface}}.$$

using Dirichlet-Neumann operators.

Asymptotic models

Asymptotic models are constructed from asymptotic expansions of the Dirichlet-Neumann operators, w. r. t. given dimensionless parameters.



$$\epsilon \equiv \frac{a}{d_1}, \quad \mu \equiv \frac{{d_1}^2}{\lambda^2}, \quad \gamma \equiv \frac{\rho_1}{\rho_2}, \quad \delta \equiv \frac{d_1}{d_2}, \quad \mathrm{bo} \equiv \frac{g(\rho_2-\rho_1)d_1^2}{\sigma}.$$

Asymptotic models : examples

Shallow water : $\mu \ll 1$.

First order : Saint-Venant system

$$\partial_t U + A[\epsilon U]\partial_x U = 0.$$

Second order : Green-Naghdi (or Serre) system

$$\partial_t U + A[\epsilon U]\partial_x U + \mu B[\epsilon U, \partial_x]\partial_x U + \mu C[\epsilon U, \partial_x]\partial_t U = 0.$$

Long wave : $\mu \ll 1, \ \epsilon = \mathcal{O}(\mu)$.

Boussinesq system :

$$\partial_t U + A_0\partial_x U + \epsilon A(U)\partial_x U + \mu B\partial_x^3 U + \mu C\partial_x^2\partial_t U = 0.$$

Moderate amplitude regime : $\mu \ll 1, \ \epsilon = \mathcal{O}(\mu^{1/2})$.

Asymptotic models : remarks

- These models are justified in the sense of consistency.[1]
  In order to fully justify these models, one should prove that they are well-posed, and that their solution remains close to the full Euler system.
- These models are not unique ! One may derive a family of models, with possibly very different behavior. Typically, when $\mu \ll 1$, one can manipulate the dispersion effects on large wavelength without modifying the precision.
- Then a question is whereas one can select a (class of) model with improved properties, such as
  - optimal frequency dispersion ;
  - well-posedness for less regular initial data ;
  - well-posedness over larger time.

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Asymptotic models : state of the art

Shallow water : $\mu \ll 1$.

First order : Saint-Venant system

$$\partial_t U + A[\epsilon U]\partial_x U = 0.$$

Well-posed, $T_{\max} \geq T/\epsilon$, *stable.* Guyenne-Lannes-Saut '10

Second order : Green-Naghdi (or Serre) system

$$\partial_t U + A[\epsilon U]\partial_x U + \mu B[\epsilon U, \partial_x]\partial_x U = 0.$$

(The original is) ill-posed. Liska-Margolin-Wendroff '95, Cotter-Holm-Percival '10

Long wave : $\mu \ll 1, \ \epsilon = \mathcal{O}(\mu)$.

Boussinesq system :

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(Some are) well-posed, $T_{\max} \geq T/\epsilon$, *stable.* Bona-Chen-Saut '04, Saut-Xu '12 ...

$\rightsquigarrow$ *justification of decoupled KdV approximation.* Bona-Colin-Lannes '04, VD'11

The aim of the talk : construct a well-posed asymptotic model in the moderate amplitude regime, and use it to describe asymptotically the behavior of the flow.

The Green-Naghdi system

The Green-Naghdi system

$$\begin{cases} \partial_t \zeta \ + \ \partial_x \Big(\dfrac{h_1 h_2}{h_1 + \gamma h_2} v \Big) \ = \ 0, \\ \partial_t \Big(v + \mu \mathcal{Q}[\zeta] v \Big) + (\gamma + \delta) \partial_x \zeta + \dfrac{\epsilon}{2} \partial_x \Big(\dfrac{{h_1}^2 - \gamma {h_2}^2}{(h_1 + \gamma h_2)^2} |v|^2 \Big) \ = \ \mu \epsilon \partial_x \big(\mathcal{R}[\zeta, v] \big), \end{cases} \tag{GN}$$

with $h_1 = 1 - \epsilon\zeta$ and $h_2 = \frac{1}{\delta} + \epsilon\zeta$ and

$$v \ \equiv \ \frac{1}{h_2(t,x)} \int_{-\frac{1}{\delta}}^{\epsilon\zeta(t,x)} \partial_x \phi_2(t,x,z) \ dz \ - \ \gamma \frac{1}{h_1(t,x)} \int_{\epsilon\zeta(t,x)}^{1} \partial_x \phi_1(t,x,z) \ dz.$$

$$\mathcal{Q}[\zeta]V \ \equiv \ \frac{-1}{3 h_1 h_2} \Big(h_1 \partial_x \Big({h_2}^3 \partial_x \big(\frac{h_1 \ V}{h_1 + \gamma h_2} \big) \Big) \ + \ \gamma h_2 \partial_x \Big({h_1}^3 \partial_x \big(\frac{h_2 \ V}{h_1 + \gamma h_2} \big) \Big) \Big),$$

$$\mathcal{R}[\zeta, V] \ \equiv \ \frac{1}{2} \Big(\Big(h_2 \partial_x \big(\frac{h_1 \ V}{h_1 + \gamma h_2} \big) \Big)^2 \ - \ \gamma \Big(h_1 \partial_x \big(\frac{h_2 \ V}{h_1 + \gamma h_2} \big) \Big)^2 \Big) + \frac{1}{3} \frac{V}{h_1 + \gamma h_2} \ \Big(\frac{h_1}{h_2} \partial_x \Big({h_2}^3 \partial_x \big(\frac{h_1 \ V}{h_1 + \gamma h_2} \big) \Big) \ - \ \gamma \frac{h_2}{h_1} \partial_x \Big({h_1}^3 \partial_x \big(\frac{h_2 \ V}{h_1 + \gamma h_2} \big) \Big) \Big).$$

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$$v \;\equiv\; \frac{1}{h_2(t,x)}\int_{-\frac{1}{\delta}}^{\epsilon\zeta(t,x)}\partial_x\phi_2(t,x,z)\;dz \;-\; \gamma\frac{1}{h_1(t,x)}\int_{\epsilon\zeta(t,x)}^{1}\partial_x\phi_1(t,x,z)\;dz.$$

$$\begin{aligned} \mathcal{Q}[\zeta]V \;&\equiv\; \tfrac{-1}{3h_1h_2}\Big(h_1\partial_x\Big({h_2}^3\partial_x\big(\tfrac{h_1\;V}{h_1+\gamma h_2}\big)\Big) \;+\; \gamma h_2\partial_x\Big({h_1}^3\partial_x\big(\tfrac{h_2\;V}{h_1+\gamma h_2}\big)\Big)\Big), \\ \mathcal{R}[\zeta,V] \;&\equiv\; \tfrac{1}{2}\Big(\Big(h_2\partial_x\big(\tfrac{h_1\;V}{h_1+\gamma h_2}\big)\Big)^2 \;-\; \gamma\Big(h_1\partial_x\big(\tfrac{h_2\;V}{h_1+\gamma h_2}\big)\Big)^2\Big) \\ &\quad +\tfrac{1}{3}\,\tfrac{V}{h_1+\gamma h_2}\;\Big(\tfrac{h_1}{h_2}\partial_x\Big({h_2}^3\partial_x\big(\tfrac{h_1\;V}{h_1+\gamma h_2}\big)\Big) \;-\; \gamma\tfrac{h_2}{h_1}\partial_x\Big({h_1}^3\partial_x\big(\tfrac{h_2\;V}{h_1+\gamma h_2}\big)\Big)\Big). \end{aligned}$$

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Consistency

The full Euler system is consistent with the Green-Naghdi model, with precision $\mathcal{O}(\mu^2)$.

Remarks :

- This extends to $3D$ case, non-flat topography, surface tension.
- Linearly well-posed. Nonlinear well-posedness is completely open.
- Unconditionally ill-posed in presence of background shear.

Construction of our model

The Serre system

$$\begin{cases} \partial_t\zeta \;+\; \partial_x\Big(\dfrac{h_1 h_2}{h_1+\gamma h_2}v\Big) \;=\; 0, \\[2ex] \partial_t\Big(v+\mu\mathcal{Q}[\zeta]v\Big) \;+\; (\gamma+\delta)\partial_x\zeta \;+\; \dfrac{\epsilon}{2}\partial_x\Big(\dfrac{{h_1}^2-\gamma {h_2}^2}{(h_1+\gamma h_2)^2}|v|^2\Big) \;=\; \mu\epsilon\partial_x\big(\mathcal{R}[\zeta,v]\big), \end{cases} \tag{S}$$

with $h_1 = 1-\epsilon\zeta$, $h_2 = \frac{1}{\delta}+\epsilon\zeta$ and

$$\mathcal{Q}[\zeta]V \;\equiv\; -a\,\partial_x^2 V + \epsilon\left(b\; V\partial_x^2\zeta + c\;(\partial_x\zeta)(\partial_x V) + d\;\partial_x(\zeta\partial_x V)\right) + \mathcal{O}(\epsilon^2),$$
$$\mathcal{R}[\zeta,V] \;\equiv\; e\;(\partial_x V)^2 \;+\; f\;\, V\partial_x^2 V \;+\; \mathcal{O}(\epsilon).$$

Introduce

$$F(\epsilon\zeta,v) \;=\; F_0(\epsilon\zeta) \;+\; \epsilon^2\; F_1(\epsilon\zeta)v^2$$
$$\mathcal{S}[\epsilon\zeta]V \;=\; (1+\kappa_1\epsilon\zeta)V \;-\; \mu\; a\;\partial_x\Big((1+\kappa_2\epsilon\zeta)\partial_x V\Big).$$

Fit the parameters, extra manipulations, withdraw $\mathcal{O}(\mu^2+\mu\epsilon^2)$ terms.

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Fit the parameters, extra manipulations, withdraw $\mathcal{O}(\mu^2+\mu\epsilon^2)$ terms.

Justification of our model

The modified Serre system

$$\begin{cases} F(\epsilon\zeta,v)\partial_t\zeta \;+\; F(\epsilon\zeta,v)\partial_x\Big(\dfrac{h_1h_2}{h_1+\gamma h_2}v\Big) \;=\; 0, \\ \mathcal{S}[\epsilon\zeta](\partial_t v+\epsilon\sigma v\partial_x v) \;+\; (\gamma+\delta)(1+\kappa_1\epsilon\zeta)\partial_x\zeta \;+\; \frac{\epsilon}{2}\partial_x\Big(\big(\frac{{h_1}^2-\gamma {h_2}^2}{(h_1+\gamma h_2)^2}-\sigma\big)|v|^2\Big) \\ \qquad\qquad = \; \mu\epsilon\,\varsigma\,\partial_x\big((\partial_x v)^2\big), \end{cases} \tag{S'}$$

with $h_1=1-\epsilon\zeta$, $h_2=\frac{1}{\delta}+\epsilon\zeta$, and

$$\mathcal{S}[\epsilon\zeta]V \;=\; (1+\kappa_1\epsilon\zeta)V \;-\; \mu\, a\, \partial_x\Big((1+\kappa_2\epsilon\zeta)\partial_x V\Big).$$

Properties of the system.

- Linearly well-posed.
  Our manipulations do not modify the dispersion relation ;
- Conditionally ($|\epsilon v_0|^2 < f(\delta,\gamma)$) linearly well-posed in presence of a background shear. The original Serre system was not !
- "Symmetric" + lower order terms

Rigorous justification of the model

Consistency

The full Euler system is consistent with (S'), with precision $\mathcal{O}(\mu^2 + \mu\epsilon^2 + \mu \mathrm{bo}^{-1})$.

Well posedness

The new model is well-posed in $X^s \equiv H^s \times H^{s+1}_\mu$ $(s > 3/2)$ over times of order $\gtrsim 1/\epsilon$.

Stability

If V satisfies (S') up to $R \in L^1([0, T/\epsilon); X^s)$, then for U_S the solution of (S') with same initial data, one has

$$\forall t \in [0, T/\epsilon], \qquad \big|V - U_S\big|_{X^s} \;\leq\; C\, |R|_{L^1([0,t);X^s)}$$

Convergence

The difference between any sufficiently smooth solution U of the full Euler system, and the solution U_S of the new model (S') with corresponding initial data, satisfies

$$\forall t \in [0, T/\epsilon], \qquad \big|U - U_S\big|_{L^\infty([0,t];X^s)} \;\leq\; C\,(\mu^2 + \mu\epsilon^2 + \mu \mathrm{bo}^{-1})t.$$

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Requires the following conditions :

$$h_1 \geq h_0 > 0, \;\; h_2 \geq h_0 > 0 \qquad \Longrightarrow \frac{1}{h_1 + \gamma h_2} \in H^s \tag{H1}$$

$$F(\epsilon\zeta, v) \geq h_0 > 0, \;\; \text{i.e. } |\epsilon v|^2 \leq f(\epsilon\zeta) \tag{H2}$$

$$1 + \epsilon\kappa_i\zeta \geq h_0 > 0 \; (i = 1, 2) \qquad \Longrightarrow \mathcal{S} \text{ is elliptic.} \tag{H3}$$

This allows to obtain energy estimates in the energy space

$$\big|(\zeta, v)\big|^2_{H^s \times H^{s+1}_\mu} \; \equiv \; \big|\zeta\big|^2_{H^s} + \big|v\big|^2_{H^s} + \mu\big|\partial_x v\big|^2_{H^s}.$$

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Remarks.

- We do not use $\epsilon \lesssim \mu^{1/2}$.
- The result extends to (small) non-flat topography, (small) surface tension.

1. Introduction
   - The full Euler system
   - Asymptotic models
2. Coupled models
   - Construction
   - Full justification
3. Scalar models
   - Unidirectional approximation
   - Decoupled approximation

The unidirectional model

Seek an approximate solution under the form

$$\partial_t\zeta + a\,\partial_x\zeta + \epsilon\, b\,\zeta\partial_x\zeta + \mu\, c\,\partial_x^2\partial_t\zeta + \epsilon^2 d\,\zeta^2\partial_x\zeta + \epsilon^3 e\,\zeta^3\partial_x\zeta + \mu\epsilon\partial_x\big(f\,\zeta\partial_x^2\zeta + g\,(\partial_x\zeta)^2\big) = 0\,,$$

$$v = F[\zeta] = \alpha\,\zeta + \epsilon\,\beta\,\zeta^2 + \mu\,\nu\,\partial_x^2\zeta + \cdots\,,$$

with precision (consistency) $\mathcal{O}(\mu^2+\epsilon^4)$.

Unidirectional scalar approximation (after Constantin-Lannes '09)

If the initial data satisfies $v(0,x) = F[\zeta(0,x)]$, then let $U_{\rm uni}=(v,\zeta)$ be defined by $v(t,x) = F[\zeta(t,x)]$ and

$$\partial_t\zeta + \partial_x\zeta + \epsilon\frac{3}{2}\frac{\delta^2-\gamma}{\gamma+\delta}\zeta\partial_x\zeta - \mu\frac{1}{6}\frac{1+\gamma\delta}{\delta(\gamma+\delta)}\partial_x^2\partial_t\zeta + \epsilon^2 d\,\zeta^2\partial_x\zeta + \epsilon^3 e\,\zeta^3\partial_x\zeta + \mu\epsilon\partial_x\big(f\,\zeta\partial_x^2\zeta + g\,(\partial_x\zeta)^2\big) = 0.$$

Then $U_{\rm uni}$ is an approximate solution, with accuracy $\mathcal{O}\big((\mu^2+\epsilon^4)t\big)$.

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Decomposition of the flow

Is it true that after a certain time, any perturbation will decompose into two waves, each one satisfying (approximately) $v = F[\zeta]$ *?*

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Numerically, yes.

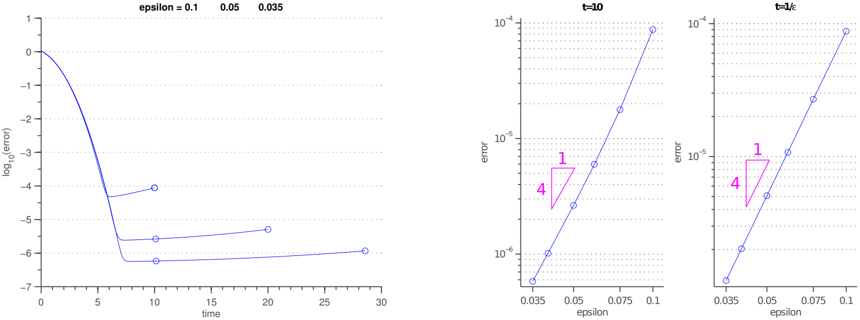

Figure : moderate amplitude regime : $\epsilon^2 = \mu$, localized initial data.

Decoupled models : Strategy

Recall the modified Serre system (S') :

$$\begin{cases} \partial_t \zeta \ + \ \partial_x\Big(\dfrac{h_1 h_2}{h_1+\gamma h_2} v\Big) \ = \ 0, \\ \mathcal{S}[\zeta](\partial_t v + \epsilon\sigma v \partial_x v) + (\gamma+\delta)(1+\kappa_1\epsilon\zeta)\partial_x\zeta + \frac{\epsilon}{2}\partial_x\Big(\big(\frac{{h_1}^2-\gamma {h_2}^2}{(h_1+\gamma h_2)^2} - \sigma\big)|v|^2\Big) \\ \qquad\qquad = \ \mu\epsilon \ \varsigma \ \partial_x\big((\partial_x v)^2\big), \end{cases}$$

1. First order : $\partial_t U + \Sigma_0 \partial_x U = 0$
   $\rightsquigarrow$ Decomposition of the flow : $U = \sum u_i \mathbf{e}_i$, $\partial_t u_i + c_i \partial_x u_i = 0$
2. Second order : WKB-type analysis
   $U_{\rm app} = \sum u_i(\iota t, t, x)\mathbf{e}_i + \iota U^c[u_i]$. $\quad \iota = \max\{\epsilon(\delta^2-\gamma), \epsilon^2, \mu\}$
   $\rightsquigarrow$ Equation on u_i, then U^c, for maximal consistency
3. Control of the secular growth of U^c.
   $\rightsquigarrow$ Consistency result.
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Rigorous justification

Well-posedness

Let $U(t=0) \in H^{s+n}$, $s > 1/2$. Then there exists a unique strong solution $u_i(\tau, t, x)$, uniformly bounded in $C^1([0,T] \times \mathbb{R}; H^{s+n})$.

The residual U^c is uniquely defined, and $U^c \in C^1([0,T] \times \mathbb{R}; H^s)$.

Secular growth of the residual

$$\forall (\tau, t) \in [0,T] \times \mathbb{R}, \quad \big| U^c(\tau, t, \cdot) \big|_{H^s} \leq C_0 \sqrt{t}.$$

Moreover, if $(1+x^2)U(t=0) \in H^{s+n}$, then one has the uniform estimate

$$\big| U^c(\tau, t, \cdot) \big|_{H^s} \leq C_0,$$

Consistency

$\sum u_i(\iota t, t, x)\mathbf{e}_i + \iota U^c(\iota t, t, x)$ satisfies the Serre model (S'), with precision $\mathcal{O}\big(\iota^2(1+\sqrt{t})\big)$ (and $\mathcal{O}(\iota^2)$ if $(1+x^2)U(t=0) \in H^{s+n}$), for $t \in [0, T/\iota]$.
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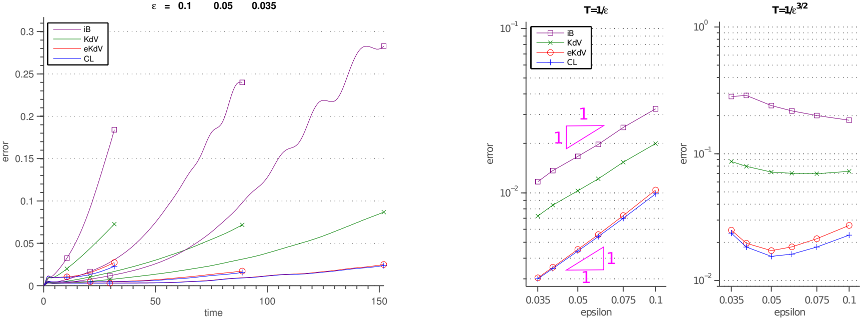
Convergence

The difference between the solution of the full Euler system and the decoupled model for $t \in [0, T/\iota]$ is of size $\mathcal{O}\big(\iota \times \min\{t, \sqrt{t}\}\big)$, and of size $\mathcal{O}\big(\iota \times \min\{t, 1\}\big)$ if the initial data is localized in space.

Error in the moderate amplitude regime

In the non-critical case $\delta^2 - \gamma \neq 0$, the inviscid Burgers' equation is as precise as any higher order decoupled model.

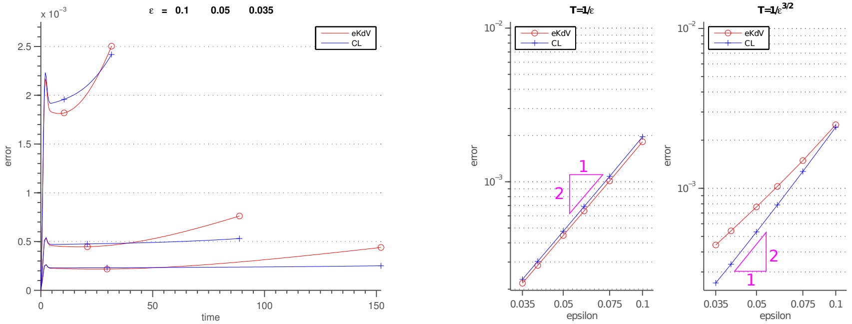
$$\partial_t u_\pm \ \pm\ \partial_x u_\pm \ \pm\ \epsilon \frac{3}{2}\frac{\delta^2-\gamma}{\gamma+\delta} u_\pm \partial_x u_\pm \ =\ 0\ . \qquad \text{(iB)}$$



Error in the moderate amplitude regime

In the critical case $\delta^2 = \gamma$, if the initial data is localized in space, then (CL) is the most precise decoupled model for very large times

$$\partial_t u_\pm \ \pm\ \partial_x u_\pm \ \pm\ \epsilon \frac{3}{2}\frac{\delta^2-\gamma}{\gamma+\delta} u_\pm \partial_x u_\pm \ -\ \mu \frac{1}{6}\frac{1+\gamma\delta}{\delta(\gamma+\delta)} \partial_x^2 \partial_t u_\pm$$
$$\pm\ \epsilon^2 d\ u_\pm^2 \partial_x u_\pm \ +\ \epsilon^3 e\ u_\pm^3 \partial_x u_\pm \ \pm\ \mu\epsilon \partial_x \big(f\ u_\pm \partial_x^2 u_\pm \ +\ g\ (\partial_x u_\pm)^2 \big) = 0. \quad \text{(CL)}$$



Thank you for your attention !