Global existence for some configurations of nearly parallel vortex filaments

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A model for nearly parallel vortex filaments

In a 3D homogeneous incompressible fluid a vortex filament is a vortex tube with infinitesimal cross section: the vorticity is a singular measure supported along a curve in $\mathbb{R}^3$.

Klein-Majda-Damodaran 95: for $N$ vortex filaments nearly parallel to $e_3$ parametrized by

$$(x_j(t, \sigma), y_j(t, \sigma), \sigma),$$

of circulation $\Gamma_j$, the evolution of $\Psi_j(t, \sigma) = x_j(t, \sigma) + iy_j(t, \sigma)$ is modeled by the 1-D Schrödinger system

$$\begin{cases} 
i \partial_t \Psi_j + \Gamma_j \partial_\sigma^2 \Psi_j + \sum_{k \neq j} \Gamma_k \frac{\Psi_j - \Psi_k}{|\Psi_j - \Psi_k|^2} = 0, & 1 \leq j \leq N. \end{cases}$$

In the case of exact parallel filaments, $\Psi_j(t, \sigma) = X_j(t)$, we get the evolution of point vortex system

$$\begin{cases} 
i \partial_t X_j + \sum_{k \neq j} \Gamma_k \frac{X_j - X_k}{|X_j - X_k|^2} = 0, & 1 \leq j \leq N. \end{cases}$$
Some results on the point vortex system dynamics

- $\Gamma_j > 0$ global existence using conservation laws,
- $N=2$, global existence since $|X_1(t) - X_2(t)|$ is conserved, $(X_1(t), X_2(t))$ rotate or translate,
- $N=3$ explicit collapse for certain configurations: shrinking turning triangle, Aref 79,
- $N=3$ vortex points placed at the vertices of an equilateral triangle rotate or translate,
- $N=3$ vortex placed at the ends and the middle of a segment, $\Gamma_j = \Gamma$, rotate or translate,
- vortex points placed at the $N \geq 4$ vertices of a regular polygon, $\Gamma_j = \Gamma$, rotate,
- also the vertices of regular polygons, with $\Gamma_j = \Gamma$, together with the center of the polygon form a relative equilibrium configuration,
- Kelvin’s conjecture 1878: the polygon configuration is stable iff $N \leq 7$, Novikov 75, Kurakin-Yudovich 02.
Results on the nearly parallel vortex filaments

On perturbations of exact parallel filaments, $\Psi_j(t, \sigma) = X_j(t) + u_j(t, \sigma)$:

- **Klein-Majda-Damodaran 95:**
  $N = 2$, the linearized system is stable if $\Gamma_1 / \Gamma_2 > 0$ and unstable if $\Gamma_1 / \Gamma_2 < 0$. Numerical computations on the perturbations suggest global existence in the first case and collision in the second.

- **Kenig-Ponce-Vega 03:**
  $\forall N$ local existence for any $(X_j(0))$ and small $H^1$ perturbations $(u_j(0))$, existence time $\gtrsim \log(\sum \|u_j(0)\|_{H^1})$.
  $N = 2$ global existence for any $(X_j(0))$, $\Gamma_j = \Gamma > 0$.
  $N = 3$ global existence for $(X_j(0))$ equilateral triangle, $\Gamma_j = \Gamma > 0$.
  The global existence proofs are based on

  $$|X_j(t) - X_k(t)| = d, \forall 1 \leq j \neq k \leq N$$

  which insures the conservation of the energy $\mathcal{E}(t)$

  $$\sum \int |\partial_\sigma \Psi_j(t, \sigma)|^2 \, d\sigma + \sum \int -\ln\left(\frac{|\Psi_{jk}(t, \sigma)|^2}{|X_{jk}(t)|^2}\right) + \left(\frac{|\Psi_{jk}(t, \sigma)|^2}{|X_{jk}(t)|^2} - 1\right) \, d\sigma.$$ 

  The solutions satisfy $\frac{3}{4} \leq \frac{|\Psi_j(t, \sigma) - \Psi_k(t, \sigma)|}{|X_j(t) - X_k(t)|} \leq \frac{5}{4}$. 

Theorem (B-M 11)

$N = 4$ global existence for $(X_j(0))$ vertices of a square centered at 0, $\Gamma_j = \Gamma > 0$ and $(\Psi_1 + \Psi_3)(0, \sigma) = (\Psi_2 + \Psi_4)(0, \sigma) = 0 \ \forall \sigma$.

$N = 4$ local existence for $(X_j(0))$ vertices of a square, $\Gamma_j = \Gamma > 0$,

existence time $\gtrsim \min \{\mathcal{E}(0)^{-\frac{1}{4}} \Sigma\|u_j(0)\|_{L^2}^{-\frac{1}{2}}, \mathcal{E}(0)^{-\frac{1}{3}}\}$ with

$\mathcal{E}(0) \lesssim \Sigma\|u_j(0)\|_{H^1}^2$.

The solutions satisfy

$$\frac{3}{4} \leq \frac{|\Psi_j(t, \sigma) - \Psi_k(t, \sigma)|}{|X_j(t) - X_k(t)|} \leq \frac{5}{4}.$$ 

- the rhombus shape are conserved since $(-\Psi_3, -\Psi_4, -\Psi_1, -\Psi_2)$ is also solution,
- for global $\exists$ the inertia centrum satisfies $\sum \Psi_j(t, \sigma) = \sum X_j(t) = 0$,
- $\forall T$ there are perturbations on $[0, T]$, with $\mathcal{E}(0) \ll 1 \sim \Sigma\|u_j(0)\|_{H^1}^2$,
- $|X_j(t) - X_k(t)|$ conserved, but not the same.
Results on the nearly parallel vortex filaments

Let \((X_j(t))\) be the vertices of a rotating regular polygon of radius 1 (with or without its center). We consider dilation-rotation type perturbations that preserve the polygonal shape \(\forall t, \sigma,\)

\[\Psi_j(t, \sigma) = X_j(t)\Phi(t, \sigma).\]

**Theorem 2 (B-M 11)**

- If \(\Phi(0) - 1\) is small in \(H^1\) then we have global existence and
  \[\frac{3}{4} \leq \frac{|\Psi_j(t, \sigma) - \Psi_k(t, \sigma)|}{|X_j(t) - X_k(t)|} = |\Phi(t, \sigma)| \leq \frac{5}{4}, \Psi_j(t, \sigma) \xrightarrow{|\sigma| \to \infty} X_j(t).\]

- If \(\mathcal{E}(0) = \frac{1}{2} \int |\partial_\sigma \Phi(0)|^2 + \frac{\omega}{2} \int (|\Phi(0)|^2 - 1 - \ln |\Phi(0)|^2)\) is small then we have global existence and
  \[\frac{3}{4} \leq \frac{|\Psi_j(t, \sigma) - \Psi_k(t, \sigma)|}{|X_j(t) - X_k(t)|} \leq \frac{5}{4}.\]

Moreover, if \(\Phi(0, \sigma) \xrightarrow{|\sigma| \to \infty} 1\) then \(\Psi_j(t, \sigma) \xrightarrow{|\sigma| \to \infty} X_j(t).\)
Results on the nearly parallel vortex filaments

- Gross-Pitaevskii type dynamics for the perturbation \( \Phi \)

\[
i \partial_t \Phi + \partial_{\sigma}^2 \Phi + \omega \frac{\Phi}{|\Phi|^2} (1 - |\Phi|^2) = 0,
\]

with \( \omega \in \mathbb{R}^+ \) the rotating speed of the point vortices,

- conservation of the energy

\[
E(t) = \frac{1}{2} \int |\partial_{\sigma} \Phi(t)|^2 + \frac{\omega}{2} \int (|\Phi(t)|^2 - 1 - \ln |\Phi(t)|^2).
\]

- the energy space contains small rotation type perturbations and grey solitons (finite energy travelling waves of G-P),

- existence of travelling waves,

- in progress: collisions,

- for shift type perturbations \( \Psi_j(t, \sigma) = X_j(t) + u(t, \sigma) \), linear Schrödinger dynamics.
Proof of Theorem 2

Lemma 1
Energy $\mathcal{E}(t)$ small enough implies $\|\|\Phi(t)|^2 - 1\|_{L^\infty} \leq \frac{1}{4}$.

The function $f(x) = x - 1 - \log x$ is positive and convex, and vanishes only at $x = 1$. If $\exists \sigma_0$ such that $|\Phi(t, \sigma_0)| > \sqrt{\frac{5}{4}}$ then

$|\Phi(t, \sigma)| \geq |\Phi(t, \sigma_0)| + \left| \int_{\sigma_0}^{\sigma} \partial_x \Phi(t, x)dx \right| \geq \sqrt{\frac{5}{4}} - \sqrt{2\mathcal{E}(\Phi(t))}|\sigma - \sigma_0|,$

and $|\Phi(t, \sigma)| > \sqrt{\frac{9}{8}}$ sur $I = [\sigma_0 - \frac{1}{500\mathcal{E}(t)}, \sigma_0 + \frac{1}{500\mathcal{E}(t)}]$. Finally,

$\mathcal{E}(t) \geq \frac{1}{2} f \left( \frac{9}{8} \right) |I| = \frac{1}{1000\mathcal{E}(t)} f \left( \frac{9}{8} \right),$

contradiction for $\mathcal{E}(t)$ small enough.

Since $\frac{1}{2}(x - 1)^2 \leq x - 1 - \ln x \leq 10(x - 1)^2$ on $[\frac{3}{4}, \frac{5}{4}]$ we have:

Lemma 2
$\|\|\Phi(t)|^2 - 1\|_{L^\infty} \leq \frac{1}{4}$ implies the comparison of the energies:

$\mathcal{E}_{GP}(t) = \frac{1}{2} \|\partial_\sigma \Phi(t)\|_{L^2}^2 + \frac{\omega}{4} \|\Phi(t)|^2 - 1\|_{L^2}^2 \leq \mathcal{E}(t) \leq 5 \mathcal{E}_{GP}(t).$
Proof of Theorem 2: resolution in $1 + H^1$

Similar arguments for Gross-Pitaevskii in $1 + H^1$ (Béthuel-Saut 99, B-Vega 08): We first solve locally the Schrödinger-type equation satisfied by $u(t) = \Phi(t) - 1$.

Since $\Phi(0) - 1$ is small in $H^1$, Lemma 2 and Gagliardo-Niremberg imply $E = E(0)$ small. Then, by Lemma 1, the quotient $\frac{1}{|\Phi(t)|^2}$ will remain uniformly bounded.

The existence time will then depend on the $H^1$ norm of $u(t)$. The $\dot{H}^1$ norm stays bounded in time by the energy, and the $L^2$ norm satisfies

$$
\partial_t \int |u(t)|^2 = \Re \omega \int \frac{1 + u(t)}{|1 + u(t)|^2} (1 - |1 + u(t)|^2) \overline{u(t)}
$$

$$
= \Re \omega \int \frac{(1 - |1 + u(t)|^2) \overline{u(t)}}{|1 + u(t)|^2} \leq |\omega| \|1 - |\Phi(t)|^2\|_{L^2} \|u(t)\|_{L^2} \leq |\omega| 2\sqrt{E} \|u(t)\|_{L^2},
$$

so $\|u(t)\|_{L^2} \lesssim t$. By re-iterating the local in time argument we get the global existence.
Proof of Theorem 2: resolution in the energy space

Similar arguments for Gross-Pitaevskii in the energy space (Zhidkov 87, Gérard 06): We solve locally in time by a fixed point argument for the operator

\[ A(w)(t) = \omega \int_0^t e^{i(t-\tau)\partial_\tau^2} \frac{e^{i\tau\partial_\sigma^2} \Phi(0)+w(\tau)}{|e^{i\tau\partial_\sigma^2} \Phi(0)+w(\tau)|^2} \left( 1 - |e^{i\tau\partial_\sigma^2} \Phi(0) + w(\tau)|^2 \right), \]

on \[ \sup_{0 \leq t \leq T} \| w(t) \|_{H^1} \leq \epsilon. \]

By Lemma 1, \( |\Phi(0)| \geq \frac{\sqrt{3}}{2} \).

On the other hand, since the symbol of \( e^{it\partial_\sigma^2} - 1 \) is \( \frac{e^{-i\tau \xi^2} - 1}{\xi} \),

\[ \| e^{i\tau\partial_\sigma^2} \Phi(0) - \Phi(0) \|_{H^1} \leq C(1 + \tau^{\frac{1}{2}}) \| \partial_\sigma \Phi(0) \|_{L^2} \leq C(1 + \tau^{\frac{1}{2}}) \sqrt{\mathcal{E}}. \]

By taking \( \epsilon, T \) small with respect to \( \mathcal{E} \), \[ \frac{1}{|e^{i\tau\partial_\sigma^2} \Phi(0)+w(\tau)|^2} \] will stay uniformly bounded.

We obtain \[ \| A(w)(t) \|_{H^1} \leq C(\epsilon) t(C + \sqrt{\mathcal{E}}), \] and we deduce the existence of a local solution for \( \epsilon, T \) small with respect to \( \mathcal{E} \).

By re-iterating the local in time argument we get the global existence.
Proof of Theorem 1: local existence K-P-V

For a perturbation \( u_{j, 0}(\sigma) = \Psi_j(0, \sigma) - X_j(0) \) small in \( H^1 \) \( \exists T^* \in ]0, \infty] \) maximal time such that on \([0, T^*] \times \mathbb{R}\)

\[
\frac{3}{4} |X_j(t) - X_k(t)| < |\Psi_j(t, \sigma) - \Psi_k(t, \sigma)| < \frac{5}{4} |X_j(t) - X_k(t)|,
\]

so for \( T \leq T^* \) the fixed point operator can be bounded by

\[
\Sigma \| A(u_j) \|_{L^\infty([0, T], H^1)} \leq \Sigma \| u_{j, 0} \|_{H^1} + C(|X_{kl}|) T \Sigma \| u_j \|_{L^\infty([0, T], H^1)}.
\]

For \( T \) small enough we obtain on \([0, T]\) a solution \((u_j)\) such that

\[
\Sigma \| u_j \|_{L^\infty([0, T], H^1)} \leq 2 \Sigma \| u_j(0) \|_{H^1}.
\]

The solution can be extended -although the \( H^1 \) norm might grow- on \([0, T^*]\) with \( |\log(\Sigma \| u_j(0) \|_{H^1})| \lesssim T^* \). For showing the global existence it is enough to get, if \( T^* \) is supposed finite, the contradiction

\[
\frac{3}{4} |X_j(T^*) - X_k(T^*)| < |\Psi_j(T^*, \sigma) - \Psi_k(T^*, \sigma)| < \frac{5}{4} |X_j(T^*) - X_k(T^*)|.
\]
Proof of Theorem 1: towards global existence K-P-V

The following quantities are conserved

\[ \mathcal{H} = \sum_j \int |\partial_\sigma \Psi_j(t, \sigma)|^2 \, d\sigma - \sum_{j \neq k} \int \ln \left( \frac{|\Psi_{jk}(t, \sigma)|^2}{|X_{jk}(t)|^2} \right) \, d\sigma, \]

\[ \mathcal{A} = \sum_j \int \left( |\Psi_j(t, \sigma)|^2 - |X_j(t)|^2 \right) \, d\sigma, \]

\[ \mathcal{T} = \sum_{j \neq k} \int \left( |\Psi_{jk}(t, \sigma)|^2 - |X_{jk}(t)|^2 \right) \, d\sigma. \]

Let

\[ \mathcal{I}(t) = \sum_{j \neq k} \int \left( \frac{|\Psi_{jk}(t)|^2}{|X_{jk}(t)|^2} - 1 \right) \, d\sigma. \]

Since \(-\ln(x) + (x - 1) \geq \frac{1}{2} (x - 1)^2 \) for \( x \in \left[ \frac{3}{4}, \frac{5}{4} \right] \), on \([0, T^*]\) we have

\[ \mathcal{E}(t) = \mathcal{H} + \mathcal{I}(t) \geq \frac{1}{2} \sum_{j \neq k} \left\| \frac{|\Psi_{jk}(t)|^2}{|X_{jk}(t)|^2} - 1 \right\|_{L^2}^2 + \sum_j \left\| \partial_\sigma \Psi_j(t) \right\|_{L^2}^2. \]
Proof of Theorem 1: towards global existence K-P-V

By Gagliardo-Niremberg, on \([0, T^*]\),

\[
\left\| \frac{|\psi_{jk}(t, \sigma)|^2}{|X_{jk}(t)|^2} - 1 \right\|_{L^\infty} \leq C \mathcal{E}(t)^{\frac{1}{2}} \left\| \frac{\psi_{jk}(t)}{|X_{jk}(t)|^\frac{1}{2}} \right\| \mathcal{E}(t)^{\frac{1}{2}} \leq C \mathcal{E}(t),
\]

so if \(\mathcal{E}(t)\) stays small enough on \([0, T^*]\) then \(|\psi_{jk}(t, \sigma)|\) is close enough to \(|X_{jk}(t)|\) such that

\[
\frac{3}{4} |X_j(T^*) - X_k(T^*)| < |\psi_j(T^*, \sigma) - \psi_k(T^*, \sigma)| < \frac{5}{4} |X_j(T^*) - X_k(T^*)|,
\]

which is the contradiction that implies the global existence.

In the (K-P-V) cases, \(|X_{jk}(t)| = d\) so \(\mathcal{E}(t) = \frac{T}{d}\) is conserved, and global existence is obtained for small \(\mathcal{E}(0)\).

Actually, in the cases of Theorem 2, \(\mathcal{E}(t) = \mathcal{E}(\Phi(t))\) is conserved and the global existence in \(1 + H^1\) can be obtained also this way.
Proof of Theorem 1

Control of $\mathcal{E}(t)$:

$$\mathcal{E}(t) = -\mathcal{H} + \frac{1}{2} \mathcal{T} - \mathcal{A} + \frac{1}{2} (\|u_1(t) + u_3(t)\|_{L^2}^2 + \|u_2(t) + u_4(t)\|_{L^2}^2).$$

$\Rightarrow$ $\mathcal{E}(t)$ conserved and implies global existence for rhombus type perturbations,

$\Rightarrow$ for general perturbations

$$|\mathcal{E}(t)| \leq |\mathcal{E}(0)|$$

$$+ t^2 \sup_{\tau \in [0,t]} |\mathcal{E}(\tau)|^{\frac{3}{2}} (\sum \|u_j(0)\|_{L^2} + t \sup_{\tau \in [0,t]} |\mathcal{E}(\tau)|^{\frac{1}{2}}),$$

so $T^* \geq \min \left\{ \frac{1}{\sqrt{\mathcal{E}(\Phi(0))} \sum \|u_j(0)\|_{L^2}}, \frac{1}{\mathcal{E}(0)^{\frac{1}{3}}} \right\}.$