On some minimization problems arising in the theory of solitary waves

Mihai MARIŞ
Institut de Mathématiques de Toulouse

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Introduction

In many cases, the important special solutions (such as solitary waves, standing waves, stationary solutions) of nonlinear dispersive PDE are obtained as solutions of a minimization problem of the form

$$(\mathcal{P}_\lambda) \qquad \text{minimize } E(u) \text{ under the constraint } Q(u) = \lambda = \textit{constant},$$

where $E =$ "energy" and $Q =$ "charge," "mass," "momentum," etc.

Moreover, if E and Q are conserved quantities for the evolution equation and any minimizing sequence for the problem $(\mathcal{P}_\lambda)$ has a convergent subsequence, by a well-known result of Cazenave and Lions it follows that the set of solutions of $(\mathcal{P}_\lambda)$ is orbitally stable.

Introduction

The aim of this talk is:

• to discuss a general method to prove the precompactness of minimizing sequences

• to present some applications to travelling waves for NLS with nonzero conditions at infinity.

Existence of minimizers

We consider the problem

$$(\mathcal{P}_\lambda) \qquad \begin{array}{l} \text{Minimize } E(u) = \displaystyle\int_{\mathbb{R}^N} F(u(x), \nabla u(x))\, dx \\ \text{under the constraint} \quad Q(u) = \displaystyle\int_{\mathbb{R}^N} G(u(x), \nabla u(x))\, dx = \lambda, \end{array}$$

where $u : \mathbb{R}^N \longrightarrow \mathbb{R}^m$ belongs to some function space $\mathcal{X}$.

Notation: $E_{min}(\lambda) = \inf\{E(u) \mid u \in \mathcal{X}, Q(u) = \lambda\}$.

Aim: Prove the (pre)compactness of minimizing sequences for $(\mathcal{P}_\lambda)$: any sequence $(u_n)_{n\geq 1} \subset \mathcal{X}$ such that $Q(u_n) = \lambda$ and $E(u_n) \longrightarrow E_{min}(\lambda)$ has a convergent subsequence.

Tool: Concentration-compactness method (P.-L. Lions, 1984).

Concentration-Compactness Principle

CC Lemma. (P.-L. Lions) Let $(f_n)_{n\geq 1} \subset L^1(\mathbb{R}^N)$ be a sequence of **nonnegative** functions such that

$$\int_{\mathbb{R}^N} f_n \, dx \longrightarrow \alpha_0 > 0 \qquad \text{as } n \longrightarrow \infty.$$

There is a subsequence $(f_{n_k})_{k\geq 1}$ that satisfies one (and only one) of the following properties:

1. Compactness: There is $(y_k)_{k\geq 1} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$ there is $R_\varepsilon < \infty$ satisfying

$$\int_{B(y_k, R_\varepsilon)} f_{n_k} \, dx \geq \alpha_0 - \varepsilon, \qquad \forall n \geq n_\varepsilon.$$

2. Vanishing: For any $R < \infty$ we have

$$\lim_{k \to \infty} \left(\sup_{y \in \mathbb{R}^N} \int_{B(y,R)} f_{n_k} \, dx \right) = 0.$$

3. Dichotomy: There is $\alpha \in (0, \alpha_0)$ and there are nonnegative functions $f_{k,1}, f_{k,2} \in L^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |f_{n_k} - f_{k,1} - f_{k,2}| \, dx \longrightarrow 0,$$

$$\int_{\mathbb{R}^N} f_{k,1} \, dx \longrightarrow \alpha \quad \text{et} \quad \int_{\mathbb{R}^N} f_{k,2} \, dx \longrightarrow \alpha_0 - \alpha,$$

$$\text{dist}(\text{supp}(f_{k,1}), \text{supp}(f_{k,2})) \longrightarrow \infty \quad \text{as } k \longrightarrow \infty.$$

In applications, one wants to rule out vanishing and dichotomy.

How to rule out vanishing ?

Lieb's Lemma Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $\nabla u \in L^p(\mathbb{R}^N)$ and $\|\nabla u\|_{L^p} \leq C$. Assume that $meas(\{x \mid |u(x)| \geq \varepsilon\}) \geq \delta > 0$. There is a constant $\alpha = \alpha(N, p, C, \delta, \varepsilon)$ and there is $x_0 \in \mathbb{R}^N$ such that

$$meas(\{x \in B(x_0, 1) \mid |u(x)| \geq \varepsilon/2\}) \geq \alpha.$$

Lions' Lemma Let $1 < p \leq \infty$, $1 \leq q < \infty$, $p^* = \frac{pN}{N-p}$ if $p < N$, $p^* = \infty$ if $p \geq N$. If $p < N$, we also assume $q \neq p^*$.
Assume that $(u_n)_{n\geq 1}$ is bounded in $L^q(\mathbb{R}^N)$ and $(|\nabla u_n|)_{n\geq 1}$ is bounded in $L^p(\mathbb{R}^N)$. If there is $R > 0$ such that

$$\lim_{n\to\infty} \left(\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |u_n|^q \, dx \right) = 0$$

then $u_n \longrightarrow 0$ in $L^r(\mathbb{R}^N)$ for any $r \in (\min(q, p^*), \max(q, p^*))$.

How to rule out vanishing ?

Remark. In the particular case when $E(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + F(u)\, dx$ and $Q(u) = \int_{\mathbb{R}^N} G(u)\, dx$, O. Lopes ('97) gave necessary and sufficient conditions for the existence of vanishing minimizing sequences.

How to rule out dichotomy?

Remarks (P.-L. Lions)

a) The function E_{min} is subadditive:

$$E_{min}(a+b) \leq E_{min}(a) + E_{min}(b).$$

Proof. Let $\varepsilon > 0$. Consider u and v with compact support such that $Q(u) = a$, $E(u) \leq E_{min}(a) + \frac{\varepsilon}{2}$, $Q(v) = b$, $E(v) \leq E_{min}(b) + \frac{\varepsilon}{2}$. Let x be such that the supports of u and $v(x+\cdot)$ are disjoint. Then $Q(u + v(x+\cdot)) = a + b$ and $E(u + v(x+\cdot)) \leq E_{min}(a) + E_{min}(b) + \varepsilon$.

b) Dichotomy cannot occur for any minimizing sequence of $(\mathcal{P}_\lambda)$ **iff** E_{min} is strictly subadditive at level λ:

$$\forall \alpha < \lambda, \quad E_{min}(\lambda) < E_{min}(\alpha) + E_{min}(\lambda - \alpha).$$

How to rule out dichotomy?

Except some cases where the functionals are homogeneous, it is difficult to compute E_{min} or to prove directly the strict subadditivity (cf. P.-L. Lions, *On some minimization problems in Mathematical Physics: how to check strict subadditivity conditions, 1989*).

O. Lopes ('91-'00) proposed an alternative method to prove the compactness of minimizing sequences for some classes of problems. This method

- is less general than concentration-compactness
- requires C^2 regularity of all involved functionals,

but

- does not require to check directly the strict subadditivity condition.

How to rule out dichotomy?

Notation: Π = affine hyperplane in $\mathbb{R}^N$

Π^+, Π^- = the two half-spaces generated by Π

s_Π = symmetry with respect to Π

u = function defined on $\mathbb{R}^N$

$$u_{\Pi^+}(x) = \begin{cases} u(x) \text{ if } x \in \Pi^+ \cup \Pi \\ u(s_\Pi(x)) \text{ if } x \in \Pi^-, \end{cases}$$

$$u_{\Pi^-}(x) = \begin{cases} u(s_\Pi(x)) \text{ if } x \in \Pi^+ \\ u(x) \text{ if } x \in \Pi^- \cup \Pi. \end{cases}$$

How to rule out dichotomy?

Proposition 1 (M.) Assume that:

- $\lim_{\lambda\to 0} E_{min}(\lambda) = E_{min}(0) = 0$.
- Functions with compact support are dense in $\mathcal{X}$.
- For any $v \in \mathcal{X}$ and any hyperplane Π we have $v_{\Pi^+}, v_{\Pi^-} \in \mathcal{X}$.
- F and G are symmetric with respect to one direction in $\mathbb{R}^N$, for instance $F(u, \xi_1, \ldots, -\xi_N) = F(u, \xi_1, \ldots, \xi_N)$ and $G(u, \xi_1, \ldots, -\xi_N) = G(u, \xi_1, \ldots, \xi_N)$.

Then:

a) E_{min} is concave.

b) If E_{min} is not strictly subadditive at level $\lambda_0 > 0$, then E_{min} is linear on $[0, \lambda_0]$ and there exists a vanishing minimizing sequence for $(\mathcal{P}_{\lambda_0})$.

How to rule out dichotomy?

Proof. a) Let $a < b$. Fix $\varepsilon > 0$. Take $u \in \mathcal{X}$ such that $Q(u) = \frac{a+b}{2}$ and $E(u) \leq E_{min}(\frac{a+b}{2}) + \varepsilon$. Choose t such that

$$\int_{\{x_N < t\}} G(u, \nabla u)\, dx = \frac{a}{2} \quad \text{and} \quad \int_{\{x_N \geq t\}} G(u, \nabla u)\, dx = \frac{b}{2}.$$

Let
$$u_1(x) = \begin{cases} u(x) \text{ if } x_N \leq t \\ u(x_1, \dots, 2t - x_N) \text{ if } x_N > t, \end{cases}$$

$$u_2(x) = \begin{cases} u(x_1, \dots, 2t - x_N) \text{ if } x_N < t \\ u(x) \text{ if } x_N \geq t. \end{cases}$$

Then $Q(u_1) = a$, $Q(u_2) = b$ and

$E_{min}(a) + E_{min}(b) \leq E(u_1) + E(u_2) = 2E(u) \leq 2E_{min}(\frac{a+b}{2}) + 2\varepsilon$.

Since ε is arbitrary $\Rightarrow E_{min}(\frac{a+b}{2}) \geq \frac{1}{2}(E_{min}(a) + E_{min}(b))$.

E_{min} is also subadditive and right continuous at $0 \Rightarrow E_{min}$ is concave.

How to rule out dichotomy?

b) If there is $\alpha \in]0, \lambda_0[$ such that $E_{min}(\lambda_0) = E_{min}(\alpha) + E_{min}(\lambda_0 - \alpha)$, the concavity of E_{min} implies that it is linear on $[0, \lambda_0]$.

Let $n \in \mathbb{N}^*$. Let $v_n \in \mathcal{X}$ be a function with compact support such that $Q(v_n) = \frac{\lambda_0}{n}$ and $E(v_n) \leq E_{min}(\frac{\lambda_0}{n}) + \frac{1}{n^2} = \frac{1}{n} E_{min}(\lambda_0) + \frac{1}{n^2}$.
Choose $x_1, \dots x_n \in \mathbb{R}^N$ "far away from each other," such that the supports of $v_n(\cdot + x_j)$ et $v_n(\cdot + x_k)$ are disjoint for $j \neq k$. Let $u_n = v_n(\cdot + x_1) + \cdots + v_n(\cdot + x_n)$.
Then $Q(u_n) = nQ(v_n) = \lambda_0$, $E(u_n) = nE(v_n) \leq E_{min}(\lambda_0) + \frac{1}{n}$ and $(u_n)_{n \geq 1}$ is a vanishing minimizing sequence. QED

Are there minimizers if E_{min} is linear?

Proposition 2

a) The same hypothesis as in the previous Proposition. Moreover, we assume that any minimizer of $(\mathcal{P}_\lambda)$ is C^1. If E_{min} is linear on $[0, \lambda_0]$, then for any $\lambda \in]0, \lambda_0]$, the problem $(\mathcal{P}_\lambda)$ does not admit minimizers.

b) Assume, in addition, that F and G present a bidimensional symmetry. If E_{min} is affine on $]a, b[$, where $a > 0$, then for any $\lambda \in]a, b[$, $(\mathcal{P}_\lambda)$ does not admit minimizers.

What about vanishing?

Remark. We make the same assumptions as in Proposition 1.

Moreover, we assume that:

- If $(u_n)_{n\geq 1}$ is a minimizing sequence and the sequences $(x_n) \subset \mathbb{R}^N$, $R_n \longrightarrow \infty$ and $h_n \longrightarrow \infty$ are such that
$$\int_{B(x_n, R_n+h_n)\setminus B(x_n,R_n)} F(u_n, \nabla u_n)\, dx \longrightarrow 0,$$
than there are $v_n, w_n \in \mathcal{X}$ such that $v_n = u_n$ on $B(x_n, R_n)$, $w_n = u_n$ on $\mathbb{R}^N \setminus B(x_n, R_n + h_n)$ and
$$E(u_n) - E(v_n) - E(w_n) \longrightarrow 0,$$
$$Q(u_n) - Q(v_n) - Q(w_n) \longrightarrow 0.$$

Then vanishing may occur for a minimizing sequence of $(\mathcal{P}_\lambda)$ **iff** E_{min} is linear on $[0, \lambda]$.

Some qualitative properties

Theorem (M. 07) Assume that:

- F and G have a common symmetry (at least bidimensional).
- For any $v \in \mathcal{X}$ and any hyperplane Π we have $v_{\Pi^+}, v_{\Pi^-} \in \mathcal{X}$.
- $(\mathcal{P}_\lambda)$ admits minimizers and any minimizer is C^1.

Then, after a translation, any minimizer has the same symmetry as F and G.

In particular, if F and G depend only on u and $|\xi|$, any minimizer is radially symmetric (after a translation in $\mathbb{R}^N$).

Travelling waves for NLS

We consider the nonlinear Schrödinger equation

$$(NLS) \qquad i\frac{\partial \Phi}{\partial t} + \Delta\Phi + F(|\Phi|^2)\Phi = 0 \qquad \text{in } \mathbb{R}^N,$$

together with the "boundary condition"

$$|\Phi| \longrightarrow r_0 > 0 \qquad \text{as } |x| \longrightarrow \infty,$$

where $F(r_0^2) = 0$ and $F'(r_0^2) < 0$.

Important particular cases:

- The Gross-Pitaevskii equation: $F(s) = 1 - s$,
- The "cubic-quintic" NLS:

$$F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2,$$

where $\alpha_1, \alpha_3, \alpha_5 > 0$ and F has 2 positive roots.

Motivation

Eq. (1), with the considered boundary conditions at infinity, arises in the modeling of a a large variety of physical phenomena, such as:

- nonlinear optics - dark solitons
- superconductivity,
- superfluidity in Helium II,
- Bose-Einstein condensate,
- phase transitions...

Hamiltonian structure

Eq. (NLS) is Hamiltonian, the **conserved energy** is

$$E(\Phi) = \int_{\mathbb{R}^N} |\nabla \Phi|^2 \, dx + \int_{\mathbb{R}^N} V(|\Phi|^2) \, dx,$$

where $V(s) = \int_s^{r_0^2} F(\tau) \, d\tau$.

Another important conserved quantity for Eq. (NLS) is the momentum

$$P(\Phi) = (P_1(\Phi), \dots, P_N(\Phi)).$$

Formally we have

$$P_k(\Phi) = \int_{\mathbb{R}^N} \langle i \frac{\partial \Phi}{\partial x_k}, \Phi \rangle \, dx.$$

The energy space

Denoting $a = \sqrt{-\frac{1}{2}F'(r_0^2)} > 0$, we have the expansion

$$V(s) = a^2(s - r_0^2)^2 + o((s - r_0^2)^2) \qquad \text{as} \qquad s \longrightarrow r_0^2.$$

Take a nondecreasing cut-off function $\varphi \in C^\infty([0, \infty))$ such that $\varphi(s) = s$ for $s \in [0, 2r_0]$, $\varphi = constant$ near infinity.

The following **Ginzburg-Landau energy** is relevant for (NLS):

$$E_{GL}(\Phi) = \int_{\mathbb{R}^N} |\nabla \Phi|^2 \, dx + a^2 \int_{\mathbb{R}^N} (\varphi(|\Phi|)^2 - r_0^2)^2 dx.$$

The function space naturally associated to (NLS) is

$$\mathcal{E} = \{\psi \in H^1_{loc}(\mathbb{R}^N) \mid \nabla \psi \in L^2(\mathbb{R}^N),\ \varphi(|\psi|)^2 - r_0^2 \in L^2(\mathbb{R}^N)\}.$$

The sound velocity associated to (NLS) is $v_s = 2ar_0$.

The momentum

The momentum (with respect to x_1) is a functional Q such that $Q'(\psi) = 2i\psi_{x_1}$.

- If $\psi = r_0 + u \in r_0 + H^1(\mathbb{R}^N)$, we should have $Q(\psi) = \int_{\mathbb{R}^N} \langle iu_{x_1}, u \rangle \, dx$.
- If $\psi \in \mathcal{E}$ has a lifting $\psi = \rho e^{i\theta}$, we have (formally)

$$Q(\psi) = -\int_{\mathbb{R}^N} \rho^2 \theta_{x_1} \, dx = -\int_{\mathbb{R}^N} (\rho^2 - r_0^2)\theta_{x_1} \, dx.$$

If $N \geq 2$, one can prove that for any $\psi \in \mathcal{E}$ we have $\langle i\psi_{x_1}, \psi \rangle \in L^1(\mathbb{R}^N) + \partial_1 \dot{H}^1(\mathbb{R}^N)$. Let $L(v + \partial_1 w) = \displaystyle\int_{\mathbb{R}^N} v \, dx$ for $v \in L^1(\mathbb{R}^N)$ and $w \in \dot{H}^1(\mathbb{R}^N)$. Then L is well-defined and is a linear form on $L^1(\mathbb{R}^N) + \partial_1 \dot{H}^1(\mathbb{R}^N)$. This enables us to define

$$Q(\psi) = L(\langle i\psi_{x_1}, \psi \rangle) \qquad \forall \psi \in \mathcal{E}.$$

One then easily checks that Q has all expected properties.

The Cauchy Problem

• F. Béthuel - J.-C. Saut '99:

Global well-posedness in $r_0 + H^1(\mathbb{R}^N)$, $N = 1, 2, 3$.

• C. Gallo '05:

Global well-posedness in Zhidkov spaces $X^k(\mathbb{R}^N)$.

• P. Gérard '06:

Global well-posedness in the energy space $\mathcal{E}$ for $N = 1, 2, 3$ as well as for $N = 4$ and initial data with small energy.

Aim: Understand the long-time dynamics associated to (NLS).

The first step in this direction is to understand the special solutions and the behavior of solutions with initial data close to the special solutions.

Travelling waves

Travelling waves of speed c are special solutions of the form $\Phi(x,t) = \psi(x_1 - ct, x_2, \dots, x_N)$. They satisfy the equation

$$-ic\psi_{x_1} + \Delta\psi + F(|\psi|^2)\psi = 0 \qquad \text{in } \mathbb{R}^N.$$

These solutions are supposed to play an important role in the long-time dynamics of (1).

Travelling waves also model various phenomena observed in Helium II such as vortices, sound waves, etc.

The existence and the qualitative properties of travelling-waves have been extensively studied numerically in the physical literature and more recently at a rigorous level.

Formally, a travelling wave of speed c is a critical point of $E - cQ$.

Minimizing the energy at fixed momentum

Consider the problem

$$(\mathcal{P}_p) \qquad \text{minimize } E(\psi) \text{ in the set } \{\psi \in \mathcal{E} \mid Q(\psi) = p\}.$$

Let $E_{min}(p) = \inf\{E(\psi) \mid \psi \in \mathcal{E}, Q(\psi) = p\}$.

If V is negative somewhere, it can be proved that for any p we have $E_{min}(p) = -\infty$. From now on we assume that $V \geq 0$ on $[0, \infty)$.

Clearly, any minimizer ψ of $(\mathcal{P}_p)$ satisfies an equation

$E'(\psi) - c(\psi)Q'(\psi) = 0 \quad \Rightarrow \psi$ is a t.w. of speed $c(\psi)$.

Previous results

For the Gross-Pitaevskii equation ($F(s) = 1 - s$), the existence of minimizers of $(\mathcal{P}_p)$ (but not the compactness of all minimizing sequences!) has been proved by F. Béthuel, P. Gravejat and J.-C. Saut (2007).

• $N = 2$: E_{min} is strictly concave on $(0, \infty)$ and $(\mathcal{P}_p)$ admits a minimizer ψ_p for any $p > 0$.

• $N = 3$: $E_{min}(p) = v_s p$ for $p \in (0, p_0]$, E_{min} is strictly concave on (p_0, ∞) and $(\mathcal{P}_p)$ admits a minimizer ψ_p for any $p > p_0$.

Moreover, $\frac{dE^+_{min}}{dp}(p) \le c(\psi_p) \le \frac{dE^-_{min}}{dp}(p)$ and $\lim_{p\to\infty} c(\psi_p) = 0$,
$\lim_{p\to 0} c(\psi_p) = v_s$ if $N = 2$, respectively
$\lim_{p\to p_0} c(\psi_p) = c_{cr} < v_s$ if $N = 3$.

Main results

We consider the following assumptions:

A1. $F \in C([0,\infty[)$, F is C^1 near r_0^2, $F(r_0^2) = 0$ and $F'(r_0^2) < 0$,

A2. There exist $q < \frac{2}{N-2}$ ($q < \infty$ if $N = 2$) and $C > 0$ such that $|F(s)| \leq Cs^q$ for $s \geq 2r_0^2$.

Theorem (M. - D. Chiron '10) Assume **A1**, **A2** and $V \geq 0$. Then:

a) If $N = 2$, the function E_{min} is positive, strictly concave and increasing on $(0,\infty)$ and $E_{min}(p) < v_s p$ for any $p > 0$.

b) If $N \geq 3$, there is $p_0 > 0$ such that $E_{min}(p) = v_s p$ on $[0, p_0]$, $E_{min}(p) < v_s p$ and E_{min} is strictly concave and increasing on (p_0, ∞).

- If $E_{min}(p) < v_s p$, any sequence $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ such that $Q(\psi_n) \longrightarrow p$ and $E(\psi_n) \longrightarrow E_{min}(p)$ has a convergent subsequence.
- If $E_{min}(p) = v_s p$, the problem $(\mathcal{P}_p)$ has no solution.

Corollary Let $\mathcal{S}_p = \{\psi \in \mathcal{E} \mid Q(\psi) = p, E(\psi) = E_{min}(p)\}$.
For any $p > p_0$ (with $p_0 = 0$ if $N = 2$) the set $\mathcal{S}_p$ is not empty, any $\psi_p \in \mathcal{S}_p$ of it is a travelling wave of speed $c(\psi_p) \in [\frac{dE^+_{min}}{dp}(p), \frac{dE^-_{min}}{dp}(p)]$ and $\mathcal{S}_p$ is **orbitally stable** under the flow of (NLS).

Moreover, $\lim_{p\to\infty} c(\psi_p) = 0$,
$\lim_{p\to 0} c(\psi_p) = v_s$ if $N = 2$, respectively
$\lim_{p\to p_0} c(\psi_p) = c_{cr} < v_s$ if $N \geq 3$.

Sketch of the proof

Step 0. One can estimate $E(\psi)$ in terms of $E_{GL}(\psi)$ and conversely. Let $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ be such that $Q(\psi_n) \longrightarrow p$ and $E(\psi_n) \longrightarrow E_{min}(p)$. Then $E_{GL}(\psi_n)$ is bounded and passing to a subsequence, we may assume that

$$E_{GL}(\psi_n) \longrightarrow \alpha_0 > 0 \quad \text{as } n \longrightarrow \infty.$$

We use the concentration-compactness principle for the functions

$$e_n = |\nabla \psi_n|^2 + a^2(\varphi(|\psi_n|^2) - r_0^2)^2 \in L^1(\mathbb{R}^N).$$

Sketch of the proof

Step 1. Using appropriate test functions we find:

- $0 \le E_{min}(p) \le v_s p$.

  Then abstract theory $\Rightarrow$ E_{min} is concave on $[0, \infty)$.
- $E_{min}(p) < v_s p$ for any $p > 0$ if $N = 2$.
- $\lim_{p \longrightarrow \infty} \frac{E_{min}(p)}{p} = 0$.

Let $p_0 = \inf\{p > 0 \mid E_{min}(p) < v_s p\}$.

Sketch of the proof

Step 2. Let $\psi \in \mathcal{E}$ such that $r_0 - \delta \le |\psi| \le r_0 + \delta$, with $\delta > 0$ small.

We have a lifting $\psi = \rho e^{i\theta}$, where $r_0 - \delta < \rho(x) < r_0 + \delta$ and:

$$|\nabla \psi|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta|^2, \quad Q(\psi) = -\int_{\mathbb{R}^N} (\rho^2 - r_0^2)\theta_{x_1}\, dx,$$

$$V(|\psi|^2) = V(\rho^2) \sim a^2(\rho^2 - r_0^2)^2 \ge (1-\varepsilon)a^2(\rho^2 - r_0^2)^2.$$

Let $\eta > 0$. Since $v_s = 2ar_0$, we may choose $\delta, \varepsilon > 0$ s.t. $v_s - \eta < 2(1-2\varepsilon)a(r_0 - \delta)$. Then

$$E(\psi) - \varepsilon E_{GL}(\psi) - (v_s - \eta)|Q(\psi)| \ge (1-\varepsilon)\int_{\mathbb{R}^N} |\nabla \rho|^2 + \rho^2 |\nabla \theta|^2\, dx$$

$$-(v_s - \eta)\int_{\mathbb{R}^N} \left|(\rho^2 - r_0^2)\theta_{x_1}\right| dx \qquad + (1-2\varepsilon)a^2 \int_{\mathbb{R}^N} (\rho^2 - r_0^2)^2\, dx$$

$$\ge \int_{\mathbb{R}^N} (1-\varepsilon)(r_0-\delta)^2|\theta_{x_1}|^2 - (v_s - \eta)\left|(\rho^2 - r_0^2)\theta_{x_1}\right| + (1-2\varepsilon)a^2(\rho^2 - r_0^2)^2\, dx$$

≥ 0 because $a^2 - 2|ab| + b^2 \ge 0$!

Sketch of the proof

We proved that $E(\psi) - (v_s - \eta)|Q(\psi)| \geq \varepsilon E_{GL}(u) > 0$ if $|\psi|$ is sufficiently close to r_0.

Aim: Prove a similar estimate for any $\psi \in \mathcal{E}$ such that $E_{GL}(\psi)$ is sufficiently small.

A regularization procedure

Aim: Get rid of small-scale topological defects of functions.

Let $\psi \in \mathcal{E}$, $h > 0$ and $\Omega \subset \mathbb{R}^N$. We consider the functional

$$G^{\psi}_{h,\Omega}(\zeta) = E^{\Omega}_{GL}(\zeta) + \frac{1}{h^2} \int_{\Omega} \varphi \left(\frac{|\zeta - \psi|^2}{32 r_0} \right) dx.$$

We show that $G^{\psi}_{h,\Omega}$ has minimizers in the set

$$\{\zeta \in \mathcal{E} \mid \zeta = \psi \text{ in } \mathbb{R}^N \setminus \Omega, \ \zeta - \psi \in H^1_0(\Omega)\}.$$

These minimizers ζ_h have remarkable properties, for instance:

- $||\zeta_h - \psi||_{L^2(\mathbb{R}^N)} \longrightarrow 0$ as $h \longrightarrow 0$,
- If $\omega \subset\subset \Omega$ we can estimate $|| \, |\zeta_h| - r_0 \, ||_{L^\infty(\omega)}$ in terms of h and $E^{\Omega}_{GL}(\psi)$. More precisely, $|| \, |\zeta_h| - r_0 \, ||_{L^\infty(\omega)}$ is arbitrarily small if $E^{\Omega}_{GL}(\psi)$ is sufficiently small.

Sketch of the proof

Using the regularization procedure, we prove

Lemma 1. Let $0 < \eta < v_s$. For any $\varepsilon \in (0, \frac{\eta}{v_s})$ there is $K > 0$ such that for any $\psi \in \mathcal{E}$ with $E_{GL}(\psi) < K$ we have

$$E(\psi) - (v_s - \eta)|Q(\psi)| > \varepsilon E_{GL}(\psi).$$

Corollary. Let $\eta > 0$. There is $p_\eta > 0$ such that

$$E_{min}(p) > (v_s - \eta)p \qquad \text{for any } p \in (0, p_\eta).$$

Sketch of the proof

Step 3. Rule out vanishing. Argue by contradiction and assume that

$$(Van) \qquad \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} E_{GL}^{B(y,1)}(\psi_n) = 0.$$

Lemma 2. If $E_{GL}(\psi_n)$ is bounded and (ψ_n) satisfies (Van), then

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |V(|\psi_n|^2) - a^2(\varphi^2(|\psi_n|) - r_0^2)^2|\, dx = 0.$$

The proof uses Lieb's Lemma.

Lemma 3. Assume that $E_{GL}(\psi_n)$ is bounded and (ψ_n) satisfies (Van). There is a sequence $h_n \longrightarrow 0$ such that for any minimizer ζ_n of $G_{h_n,\mathbb{R}^N}^{\psi_n}$ in $H^1_{\psi_n}(\mathbb{R}^N)$ we have $\| |\zeta_n| - r_0 \|_{L^\infty(\mathbb{R}^N)} \longrightarrow 0$ as $n \longrightarrow \infty$.

The proof of Lemma 3 is difficult.

Sketch of the proof

Let h_n and ζ_n be as in Lemma 3. Let $\delta_n = \| |\zeta_n| - r_0 \|_{L^\infty(\mathbb{R}^N)} \longrightarrow 0$.

Then $E_{GL}(\zeta_n) \geq 2a(r_0 - \delta_n)|Q(\zeta_n)|$.

It is easy to prove that $|Q(\psi_n) - Q(\zeta_n)| \longrightarrow 0$, hence $Q(\zeta_n) \longrightarrow p$.

We have:

$$\begin{aligned} E(\psi_n) &= E_{GL}(\psi_n) + \int_{\mathbb{R}^N} V(|\psi_n|^2) - a^2(\varphi^2(|\psi_n|) - r_0)^2 \, dx \\ &\geq E_{GL}(\zeta_n) + \int_{\mathbb{R}^N} V(|\psi_n|^2) - a^2(\varphi^2(|\psi_n|) - r_0)^2 \, dx \\ &\geq 2a(r_0 - \delta_n)|Q(\zeta_n)| + \int_{\mathbb{R}^N} V(|\psi_n|^2) - a^2(\varphi^2(|\psi_n|) - r_0)^2 \, dx. \end{aligned}$$

Passing to the limit in the above inequality we get

$$\liminf_{n\to\infty} E(\psi_n) \geq 2ar_0 p = v_s p,$$

a contradiction. This rules out vanishing.

Sketch of the proof

Step 4. Eliminate the dichotomy.

Assume that dichotomy holds for a subsequence (still denoted (ψ_n)). Using the regularization procedure we show that there are two functions $\psi_{n,1}$, $\psi_{n,2}$ such that

$$E_{GL}(\psi_{n,1}) \longrightarrow \alpha \qquad \text{and} \qquad E_{GL}(\psi_{n,2}) \longrightarrow \alpha_0 - \alpha,$$

$$|E(\psi_n) - E(\psi_{n,1}) - E(\psi_{n,2})| \longrightarrow 0,$$

$$|Q(\psi_n) - Q(\psi_{n,1}) - Q(\psi_{n,2})| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

It is easy to see that $(Q(\psi_{n,i}))_{n\geq 1}$ are bounded, $i = 1, 2$. Passing again to a subsequence, we may assume that

$$Q(\psi_{n,1}) \longrightarrow p_1 \quad \text{and} \quad Q(\psi_{n,2}) \longrightarrow p_2 \quad \text{where } p_1 + p_2 = p.$$

Sketch of the proof

It is easy to see that $p_i \in (0, p)$, $i = 1, 2$. Now abstract theory (E_{min} concave) $\Rightarrow E_{min}$ is linear on $[0, p]$, say $E_{min}(s) = \kappa s$. We know that $\kappa \leq v_s$. Lemma 1 implies $\kappa > v_s - \eta$ for any $\eta > 0$, thus $\kappa = v_s$ and $E_{min}(p) = v_s p$, a contradiction.

Hence dichotomy cannot occur.

Sketch of the proof

Step 5. Conclusion.

Vanishing and dichotomy are excluded, thus we have compactness: there exists a sequence $(x_n)_{n\geq 1} \subset \mathbb{R}^N$ such that, denoting $\tilde{\psi}_n = \psi_n(\cdot + x_n)$, we have:

$$\text{for all } \varepsilon > 0, \text{ there exists } R_\varepsilon > 0 \text{ and } n_\varepsilon \in \mathbb{N}^* \text{ such that}$$
$$E_{GL}^{\mathbb{R}^N \setminus B(0,R_\varepsilon)}(\tilde{\psi}_n) < \varepsilon \text{ for any } n \geq n_\varepsilon.$$

We prove that there is a subsequence $(\tilde{\psi}_{n_k})_{k\geq 1}$ and $\psi \in \mathcal{E}$ such that

$$\begin{array}{l} \nabla \tilde{\psi}_{n_k} \longrightarrow \nabla \psi \\ \varphi^2(|\tilde{\psi}_{n_k}|) - r_0^2 \longrightarrow \varphi^2(|\psi|) - r_0^2 \end{array} \quad \text{in } L^2(\mathbb{R}^N).$$

Then we prove that $Q(\psi) = p$ and $E(\psi) = E_{min}(p)$, hence ψ solves $(\mathcal{P}_p)$.

Sketch of the proof

It is standard (Implicit Function Theorem) to prove that any minimizer of $(\mathcal{P}_p)$ satisfies an Euler-Lagrange equation

$$E'(\psi_p) - c(\psi_p)Q'(\psi_p) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

where $c(\psi_p) \in [\frac{dE^+_{min}}{dp}(p), \frac{dE^-_{min}}{dp}(p)]$.
Elliptic regularity $\Rightarrow \psi_p \in W^{2,q}_{loc}(\mathbb{R}^N)$ and
$\nabla u \in W^{1,q}(\mathbb{R}^N)$, $\forall q \in [2, \infty[$.
(If $F \in C^k$, then $\psi_p \in W^{k+2,q}_{loc}(\mathbb{R}^N)$ and
$\nabla u \in W^{k+1,q}(\mathbb{R}^N)$, $\forall q \in [2, \infty[$.)
Symmetry theory $\Rightarrow \psi_p$ is symmetric with respect to Ox_1 if $N \geq 3$.
If $N \geq 3$ we prove that (NLS) does not admit small energy travelling waves $\Rightarrow p_0 > 0$.

QED

Open problems

Let $p > p_0$. Let $p_n \uparrow p$ and let ψ_{p_n} be a minimizer of $(\mathcal{P}_{p_n})$. Then $(\psi_{p_n})_{n\geq 1}$ is a minimizing sequence for $(\mathcal{P}_p)$, hence it converges to a minimizer $\psi_{p,1}$ of $(\mathcal{P}_p)$.

It can be proved that $c(\psi_{p,1}) = \lim_{n\to\infty} c(\psi_{p_n}) = \frac{dE^-_{min}}{dp}(p)$.

Similarly, there is a minimizer $\psi_{p,2}$ of $(\mathcal{P}_p)$ such that $c(\psi_{p,2}) = \frac{dE^+_{min}}{dp}(p)$.

Problem. a) Is it true that E_{min} is differentiable at any point $p > p_0$?
b) Is it true that $\mathcal{S}_p$ ($=$ the set of solutions of $(\mathcal{P}_p)$) is, in some sense, connected?

A positive answer to this question would imply that for any speed $c \in (0, c_{cr})$ (where $c_{cr} = \frac{dE^+_{min}}{dp}(p_0)$) there are travelling waves for (NLS) that minimize the energy at fixed momentum.

Open problems

Assume that F and V satisfy assumptions **A1**, **A2**, $V \geq 0$ on $[0, \infty)$ and there is some s_0 far away from r_0 such that $V(s_0) = 0$.

Let $\tilde{F}$ and $\tilde{V}$ be such that $F = \tilde{F}$ and $V = \tilde{V}$ except on a small neighborhood of s_0, $\|F - \tilde{F}\|_{L^\infty}$ and $\|V - \tilde{V}\|_{L^\infty}$ are small, but $\tilde{V}$ achieves negative values.

Let $\tilde{E}(\psi) = \int_{\mathbb{R}^N} |\nabla \psi|^2 \, dx + \int_{\mathbb{R}^N} \tilde{V}(|\psi|^2) \, dx.$

It is easy to prove that

$$\tilde{E}_{min}(p) = \inf\{\tilde{E}(\psi) \mid \psi \in \mathcal{E}, Q(\psi) = p\} = -\infty \text{ for any } p.$$

Problem. a) Is there an open set $\mathcal{O} \subset \mathcal{E}$ such that $\tilde{E}$ admits minimizers in $\{\psi \in \mathcal{O} \mid Q(\psi) = p\}$?

b) Is the answer to the above question affirmative for any F and V satisfying **A1** and **A2**?

Open problems

Notice that if F and V satisfy **A1** and **A2** and $N \geq 3$, it has been proved that for any $c \in (0, v_s)$, (NLS) admits travelling waves of speed c that minimize the action $E - cQ$ under a Pohozaev constraint.

Thank you very much for your attention!