An optimal adaptive finite element method

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Optimal adaptive finite element methods

Model problem: Poisson, 2D, newest vertex bisection.
[Generalizations: $\nabla \cdot \mathbf{A} \nabla$ with $\mathbf{A}$ symm. pos. def. piecewise constant, any space dimension n, red-refinements.]

Given $f \in H^{-1}(\Omega)$, find $u \in H_0^1(\Omega)$,

$$\boxed{a(u,v)} := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v =: \boxed{f(v)} \quad (v \in H_0^1(\Omega)).$$

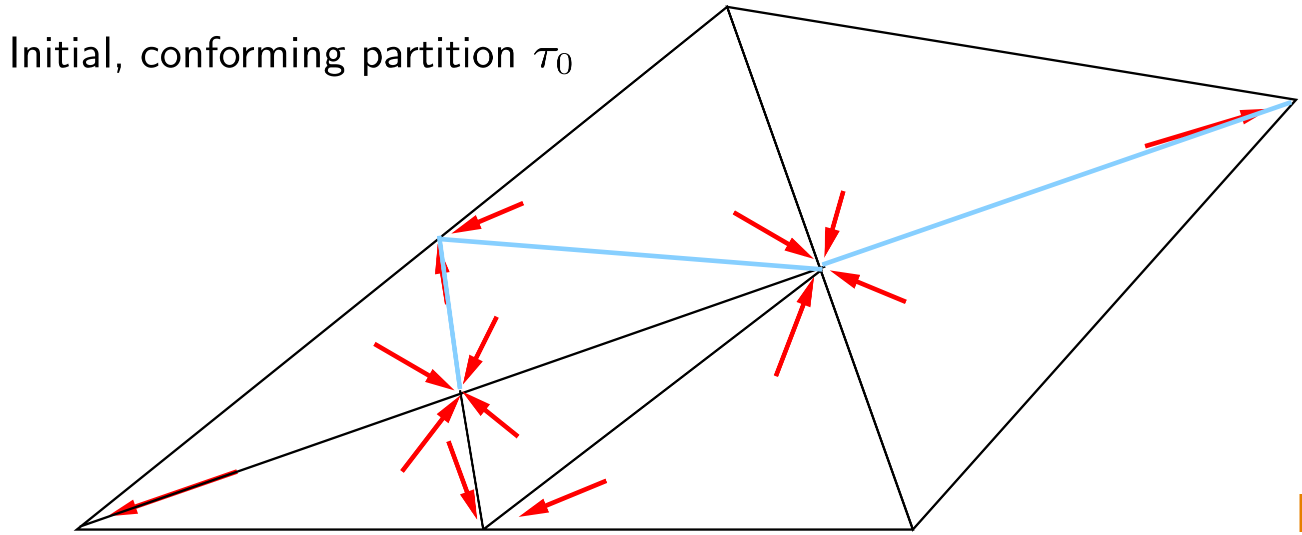
$|||\cdot||| := a(\cdot,\cdot)^{\frac{1}{2}}$.

τ conforming part. of Ω into triangles, $V_\tau = H_0^1(\Omega) \cap \prod_{T \in \tau} P_{d-1}(T)$. Find $u_\tau \in V_\tau$,

$$a(u_\tau, v_\tau) = f(v_\tau) \quad (v_\tau \in V_\tau).$$

AFEM: **GALSOLVE**, compute a post. error est, **MARK**, **REFINE**.

Newest vertex bisection



$\tau := \mathbf{REFINE}[\tau, M]$: Bisect all $T \in M \subset \tau$ a few times. Complete.

Th 1 (Binev, Dahmen, DeVore '04). *With suitable assignment of newest vertices in initial mesh,*

$$\#\tau_K - \#\tau_0 \lesssim \sum_{i=0}^{K-1} \#M_i \quad (\textit{unif. in } K).$$

(can be generalized to any space dimension [St.'08])

Perspective: Approximation classes

$$\mathcal{A}^s_\infty := \{u \in H^1_0(\Omega) : |u|_{\mathcal{A}^s_\infty} := \sup_N N^s \inf_{\{\tau\in\mathcal{P}:\#\tau-\#\tau_0\le N\}} \|u-u_\tau\|_{H^1} < \infty\},$$

[i.e. $\|u-u_\tau\|_{H^1} \le \varepsilon$ generally requires $\#\tau - \#\tau_0 \le \varepsilon^{-1/s}|u|^{1/s}_{\mathcal{A}^s_\infty}$.]

When $\mathcal{P} \sim$ **unif. refs**, then

$$u \in \mathcal{A}^{(d-1)/n}_\infty \Longleftarrow u \in H^d(\Omega), \text{ i.e., } \partial^\alpha u \in L_{\mathbf{2}}(\Omega),\ \forall|\alpha| \le d$$

With $\mathcal{P} \sim$ **all partitions created by newest vertex bisection**:

$$u \in \mathcal{A}^{(d-1)/n}_\infty \Longleftarrow u \in B^d_q(L_p(\Omega)), \text{ any } p > (\tfrac{d-1}{n} + \tfrac{1}{2})^{-1} \text{ i.e., } \partial^\alpha u \in L_{\boldsymbol{p}}(\Omega)\ \forall|\alpha| \le d$$

([Binev, Dahmen, DeVore, Petrushev '02])

Regul. th: For suff sm f, $n = 2$, u in such a Besov space for any d ([Dahlke, DeVore '97]).

A posteriori error estimator

$\eta_T(f,u_\tau) := \operatorname{diam}(T)^2\|f+\triangle u_\tau\|^2_{L_2(T)} + \operatorname{diam}(T)\|[\![\nabla u_\tau \cdot \mathbf{n}]\!]_{\partial T}\|^2_{L_2(\partial T)}$,

Th 2 (Babuška, Rheinboldt '78; Verfürth '96)**.**

$$|||u-u_\tau||| \leq C_1\mathcal{E}(\tau,f,u_\tau) := C_1[\sum_{T\in\tau}\eta_T(f,u_\tau)]^{\frac{1}{2}}.$$

Proof. $|||u-u_\tau||| = \sup_{v\in H^1_0(\Omega)} \frac{a(u-u_\tau,v)}{|||v|||}$.

$$a(u-u_\tau,v) = a(u-u_\tau,v-v_\tau) = \int_\Omega f(v-v_\tau) - a(u_\tau,v-v_\tau)$$

$$= \sum_{T\in\tau}\int_T (f+\triangle u_\tau)(v-v_\tau) - \int_{\partial T}\nabla u_\tau\cdot\mathbf{n}(v-v_\tau). \qquad \square$$

Th 3 (St.'06, Cascon,Kreuzer,Nochetto,Siebert '07)**.**

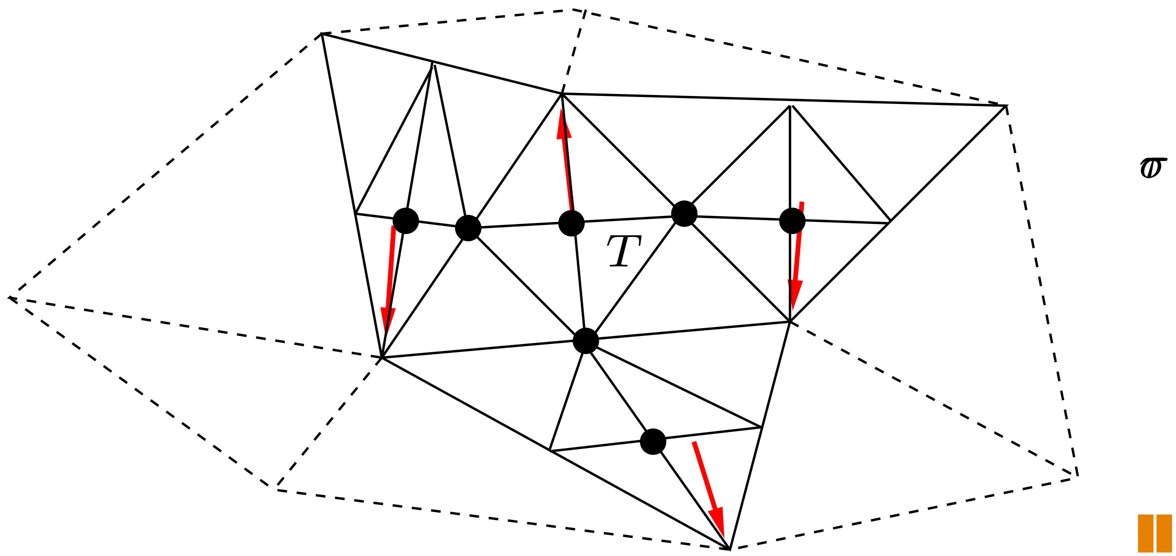
$\sigma\supset\tau$, $R_{\tau\to\sigma} := \{T\in\tau : T\notin\sigma\}$,

$$|||u_\sigma-u_\tau||| \leq C_1[\sum_{T\in R_{\tau\to\sigma}}\eta_T(f,u_\tau)]^{\frac{1}{2}}.$$

Note $\#R_{\tau\to\sigma} \lesssim \#\sigma - \#\tau$.

We call $\sigma \supset \tau$ **a full refinement with respect to $T \in \tau$**, when

T and all its neighbours in τ, as well as all faces of T contain a vertex of σ in their interiors.



Th 4 (Morin, Nochetto, Siebert '00). *Let $f \in \prod_{T \in \tau} P_{d-2}(T)$, $\sigma \supset \tau$ full ref w.r.t. $T \in \tau$. Then*

$$\eta_T(f, u_\tau) \lesssim \sum_{\tilde{T} \in \{T\} \cup \{neighbours\}} |u_\sigma - u_\tau|^2_{H^1(\tilde{T})}$$

Corol 5. *$\sigma \supset \tau$ full ref w.r.t. $T \in M \subset \tau$. Then*

$$c_2[\sum_{T\in M} \eta_T(f,u_\tau)]^{\frac{1}{2}} \leq |||u_\sigma - u_\tau|||$$ (**not true for any $f \in L_2(\Omega)$**)

In part, $c_2\mathcal{E}(\tau, f, u_\tau) \leq |||u - u_\tau|||$.

AFEM converges (Dörfler '96, Morin, Nochetto, Siebert '00)

$\mathbf{MARK}[\tau, f, u_\tau] \to M$: *Let $\theta \in (0,1]$. Select* **smallest** *$M \subset \tau$ s.t.*

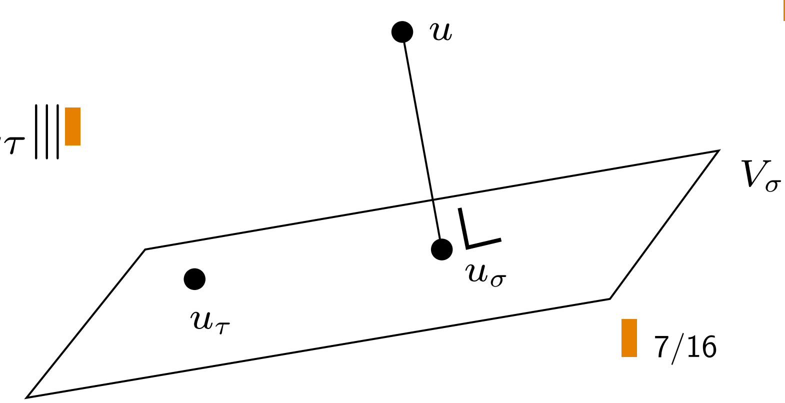
$$[\sum_{T\in M} \eta_T(\tau, f, u_\tau)]^{\frac{1}{2}} \geq \theta\, \mathcal{E}(\tau, f, u_\tau).$$

$\mathbf{REFINE}[\tau, M] \to \sigma$: *Construct smallest (conforming) $\sigma \supset \tau$ that is a full refinement w.r.t. all $T \in M$.*

$$|||u - u_\tau|||^2 = |||u - u_\sigma|||^2 + |||u_\sigma - u_\tau|||^2$$

$$|||u_\sigma - u_\tau||| \geq c_2\theta\mathcal{E}(\tau, f, u_\tau) \geq \frac{c_2\theta}{C_1}|||u - u_\tau|||$$

$$\rightsquigarrow |||u - u_\sigma||| \leq (1 - \frac{c_2^2\theta^2}{C_1^2})^{\frac{1}{2}}|||u - u_\tau|||$$



AFEM converges with optimal rate

[St '06] ([Binev,Dahmen,DeVore '04] using coarsening)

AFEM$[f,\varepsilon] \to [\tau_m, u_{\tau_m}]$ % *let* $f \in \prod_{T\in\tau_0} P_{d-2}(T)$

compute Gal sol $u_{\tau_0} \in V_{\tau_0}$; $k := 0$

`while` $C_1\mathcal{E}(\tau_k, f, u_{\tau_k}) > \varepsilon$ `do`

$M_k := \mathbf{MARK}[\tau_k, f, u_{\tau_k}]$

$\tau_{k+1} := \mathbf{REFINE}[\tau_k, M_k]$

compute Gal sol $u_{\tau_{k+1}} \in V_{\tau_{k+1}}$

$k := k+1$

`end do`

$m := k$

$$\rightsquigarrow |||u - u_{\tau_m}||| \leq \varepsilon, \quad \#\tau_m - \#\tau_0 \lesssim \sum_{k=0}^{m-1} \#M_k$$

Th 6. *If* $\theta < \frac{c_2}{C_1}$, *then*

$$\#M_k \leq \inf\left\{\#\rho - \#\tau_0 : |||u - u_\rho||| \leq [1 - \frac{C_1^2\theta^2}{c_2^2}]^{\frac{1}{2}} |||u - u_{\tau_k}|||\right\}$$

$$\leq \left[[1 - \frac{C_1^2\theta^2}{c_2^2}]^{\frac{1}{2}} |||u - u_{\tau_k}|||\right]^{-1/s} |u|_{\mathcal{A}_\infty^s}^{1/s} \quad \textit{when } u \in \mathcal{A}_\infty^s.$$

$$\rightsquigarrow \#\tau_m - \#\tau_0 \lesssim \sum_{k=0}^{m-1} |||u - u_{\tau_k}|||^{-1/s} |u|_{\mathcal{A}_\infty^s}^{1/s} \lesssim |||u - u_{\tau_{m-1}}|||^{-1/s} |u|_{\mathcal{A}_\infty^s}^{1/s} \lesssim \varepsilon^{-1/s} |u|_{\mathcal{A}_\infty^s}^{1/s}.$$

Pr. Th. 6. Let $\sigma \supset \tau_k$ s.t. $|||u - u_\sigma||| \leq [1 - \frac{C_1^2\theta^2}{c_2^2}]^{\frac{1}{2}}|||u - u_{\tau_k}|||$. Th.3 shows

$$C_1^2 \sum_{T \in R_{\tau_k \to \sigma}} \eta_T(f, u_{\tau_k}) \geq |||u_\sigma - u_{\tau_k}|||^2 = |||u - u_{\tau_k}|||^2 - |||u - u_\sigma|||^2$$

$$\geq \frac{C_1^2\theta^2}{c_2^2}|||u - u_{\tau_k}|||^2 \geq C_1^2\theta^2\mathcal{E}(\tau_k, f, u_{\tau_k})^2,$$

i.e., $[\sum_{T \in R_{\tau_k \to \sigma}} \eta_T(f, u_{\tau_k})]^{\frac{1}{2}} \geq \theta\mathcal{E}(\tau_k, f, u_{\tau_k})$.

$M := \mathbf{MARK}[\tau_k, f, u_{\tau_k}]$ is smallest set with this prop, so

$$\#M \leq \#R_{\tau_k \to \sigma} \leq \#\sigma - \#\tau_k.$$

For arb. ρ with $|||u - u_\rho||| \leq [1 - \frac{C_1^2\theta^2}{c_2^2}]^{\frac{1}{2}}|||u - u_{\tau_k}|||$, take $\sigma := \tau_k \cup \rho$. Then

$$\#M \leq \#\sigma - \#\tau_k \leq \#\rho - \#\tau_0.$$

□

General right hand sides and inexact solves

$\mathbf{RHS}[\tau, f, \delta] \to [\sigma, f_\sigma]$
% In: τ a partition, $f \in H^{-1}(\Omega)$ and $\delta > 0$.
% Out: $f_\sigma \in \prod_{T \in \tau} P_{d-2}(T)$, where $\sigma = \tau$, or, if necessary, $\sigma \supsetneq \tau$,
% such that $\|f - f_\sigma\|_{H^{-1}(\Omega)} \leq \delta$.

If $u \in \mathcal{A}^s_\infty$, cost of **RHS** will not dominate if $\exists c_f$ s.t. $\#\sigma - \#\tau \leq c_f^{1/s} \delta^{-1/s}$, and cost $\lesssim \#\sigma$. Such a pair $(f, \mathbf{RHS})$ is called s-optimal.
For suff. smooth f, s-optimality with $s = \frac{d}{n}$ $(> \frac{d-1}{n})$ can be realized.

$\mathbf{GALSOLVE}[\tau, f_\tau, u_\tau^{(0)}, \delta] \to w_\tau$
% In: τ, $f_\tau \in (\mathcal{S}_\tau)'$, and $u_\tau^{(0)} \in \mathcal{S}_\tau$.
% Out: $w_\tau \in \mathcal{S}_\tau$ with $\|u_\tau - w_\tau\|_{H^1(\Omega)} \leq \delta$.
% The call should require $\lesssim \max\{1, \log(\delta^{-1}\|u_\tau - u_\tau^{(0)}\|_{H^1(\Omega)})\}\#\tau$ ops.

AFEM has opt compl

$\mathbf{AFEM}[f,\varepsilon] \to [\tau_m, w_{\tau_m}]$
% $\omega, \beta > 0$ suff small constants.
$w_{\tau_0} := 0$; $k := 0$; $\delta_0 \eqsim \|f\|_{H^{-1}(\Omega)}$
`do`
    `do` $\delta_k := \delta_k/2$
        $[\tau_k, f_{\tau_k}] := \mathbf{RHS}[\tau_k, f, \delta_k/2]$
        $w_{\tau_k} := \mathbf{GALSOLVE}[\tau_k, f_{\tau_k}, w_{\tau_k}, \delta_k/2]$
        `if` $\eta_k := (2 + C_1 c_2^{-1})\delta_k/2 + C_1\mathcal{E}(\tau_k, f_{\tau_k}, w_{\tau_k}) \leq \varepsilon$ `then stop`
        `endif`
    `until` $\delta_k \leq \omega\mathcal{E}(\tau_k, f_{\tau_k}, w_{\tau_k})$.
    $M_k := \mathbf{MARK}[\tau_k, f_{\tau_k}, w_{\tau_k}]$
    $\tau_{k+1} := \mathbf{REFINE}[\tau_k, M_k]$
    $w_{\tau_{k+1}} := w_{\tau_k}$, $\delta_{k+1} := 2\beta\eta_k$, $k := k+1$
`enddo`
$m := k$

Th 7. $\|u - w_{\tau_m}\|_{H^1(\Omega)} \leq \varepsilon$. *If* $u \in \mathcal{A}^s_\infty$, *and* $(f, \mathbf{RHS})$ *is s-optimal, then both* $\#\tau_m$ *and work* $\lesssim \varepsilon^{-1/s}(|u|^{1/s}_{\mathcal{A}^s_\infty} + c_f^{1/s})$.

Realizations of RHS

ex. $f \in L_2(\Omega)$. Q_σ being L_2-orth.proj. onto $\prod_{T\in\sigma} P_{d-2}(T)$.

$$\|f - Q_\sigma f\|_{H^{-1}(\Omega)} \lesssim \sqrt{\sum_{T\in\sigma} \mathrm{vol}(T)^{\frac{2}{n}} \|f - Q_T f\|^2_{L_2(T)}} =: \mathrm{osc}(f,\sigma)$$

Given $\bar{\theta} \in (0,1)$, run **MARK** and **REFINE** algorithm on osc until $\leq \delta$. Is quasi-optimal in the sense that whenever

$$f \in \bar{\mathcal{A}}^s := \{f \in L_2(\Omega) : \sup_N N^s \inf_{\{\tau:\#\tau-\#\tau_0\leq N\}} \mathrm{osc}(f,\tau) < \infty\},$$

then $(f, \mathbf{RHS})$ is s-optimal (assuming $\int_T f P_{d-2}$ exact in $\mathcal{O}(1)$ operations)
So greedy works (thanks to factor $\mathrm{vol}(T)^{\frac{2}{n}}$).

ex. General d, $n > 1$. $(\frac{1}{2} + \frac{1}{n})^{-1} < q \leq 2$, $f \in L_q(\Omega)$ ($\hookrightarrow H^{-1}(\Omega)$).

Generalization of previous case with

$$\operatorname{osc}(f,\sigma) := \sqrt{\sum_{T\in\sigma} \operatorname{vol}(T)^{1-\frac{2}{q}+\frac{2}{n}} \inf_{p\in P_{d-2}(T)} \|f-p\|^2_{L_q(T)}}.$$

ex. $n = 2$, $d = 2$, $f(v) = \int_K v$, K a smooth curve. $\mathbf{RHS}[\tau, f, \delta] \to [\sigma, f_\sigma]$: Refine those T that intersect K until their diameters $\lesssim \delta^2$. $(f_\sigma)|_T := \frac{\operatorname{length}(K\cap T)}{\operatorname{vol}(T)}$. s-optimal for $s = \frac{1}{2}$ ($=\frac{d-1}{n}$).

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Extensions

- [Cascon,Kreuzer,Nochetto,Siebert '07]: One bisection of marked cells suffices (no interior node). Marking for reducing osc can be omitted, assuming $f \in L_2(\Omega)$, exact integration, and with exact solving of Gal systems.

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