A Posteriori Error Analysis for Discontinuous Galerkin Methods

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Outline

- Elliptic PDE and *hp*-FEM/DGFEM discretizations
- Stability and a priori results, exponential convergence
- A posteriori error analysis and *hp*-adaptivity
- Applications
- Summary / Future Work

Part I

hp-DGFEM, A Priori Results

Linear Elliptic PDE

- On a bounded polygon $\Omega \subset \mathbb{R}^2$, consider

$$\begin{aligned} \mathrm{L}u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $f \in L^2(\Omega)$, and L is a second-order linear elliptic operator on a space $V = H_0^1(\Omega)$, i.e.,

$$(\mathrm{L}u, v) = a(u, v) \qquad u, v \in V,$$

with

$$a(u, u) \geq C_1 \|u\|_V^2, \qquad |a(u, v)| \leq C_2 \|u\|_V \|v\|_V$$

for all $u, v \in V$.

- Variational Formulation: Find $u \in V$ such that

$$a(u, v) = \ell(v) \qquad \forall v \in V.$$

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hp-FEM Discretization

- Standard hp-finite element space:

$$V_{FEM} = \left\{ v \in H^1_0(\Omega) : v|_K \in \mathcal{S}_{p_K}(K), K \in \mathcal{T} \right\}.$$

- hp-FEM: Restriction of the continuous variational formulation to the finite element space $V_{FEM} \subset V$: Find $u_{FEM} \in V_{FEM}$ such that

$$a(u_{FEM}, v) = \ell(v)$$

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hp-DGFEM

- *hp*-DG finite element space:

$$V_{DG} = \{v \in L^2(\Omega) : v|_K \in \mathcal{S}_{p_K}(K), K \in \mathcal{T}\},$$

- *hp*-DGFEM: Find $u_{DG} \in V_{DG}$ such that

$$a_{DG}(u_{DG}, v) = \ell_{DG}(v) \qquad \forall v \in V_{DG},$$

with

$$a_{DG}(w, v) = \sum_{K \in \mathcal{T}} a_K(w, v) + F_{DG}(w, v)$$

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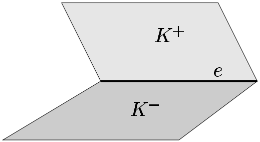
hp-DGFEM

- Trace operators:

  Jumps: $[\![v]\!] = (v^+ - v^-)\,\boldsymbol{\nu}$

  Averages: $\{\!\{v\}\!\} = \frac{1}{2}(v^+ + v^-)$



- DG inner product and norm:

$$(w, v)_{DG} = \int_\Omega \nabla_h w \cdot \nabla_h v \,\mathrm{d}\mathbf{x} + \gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 [\![w]\!][\![v]\!] \,\mathrm{d}s.$$

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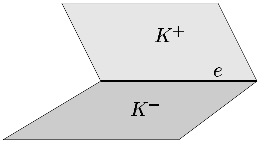

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- Example: *hp*-IP-DG discretization of $\mathrm{L} = -\Delta$.

$$
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a(u,v) = \int_\Omega \nabla_h u \cdot \nabla_h v \,\mathrm{d}\mathbf{x} - \int_{\mathcal{E}} \{\!\{\nabla_h u\}\!\} \cdot [\![v]\!] \,\mathrm{d}s \\
+\theta \int_{\mathcal{E}} \{\!\{\nabla_h v\}\!\} \cdot [\![u]\!] \,\mathrm{d}s \\
+\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \,[\![u]\!] \cdot [\![v]\!] \,\mathrm{d}s
\end{aligned}
$$

$\theta \in [-1,1]$, $\gamma > 0$ sufficiently large.

- Stability: For $\gamma > 0$ sufficiently large, there holds

$$
a_{DG}(u,u) \ge C_1 \|u\|_{DG}^2, \qquad |a_{DG}(u,v)| \le C_2 \|u\|_{DG} \|v\|_{DG}
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for all $u, v \in V_{DG}$.

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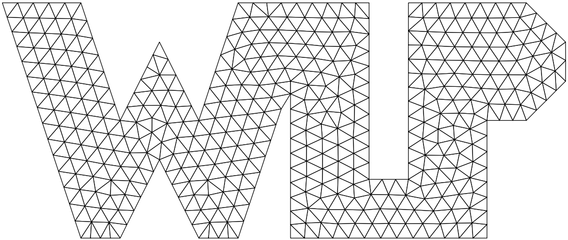
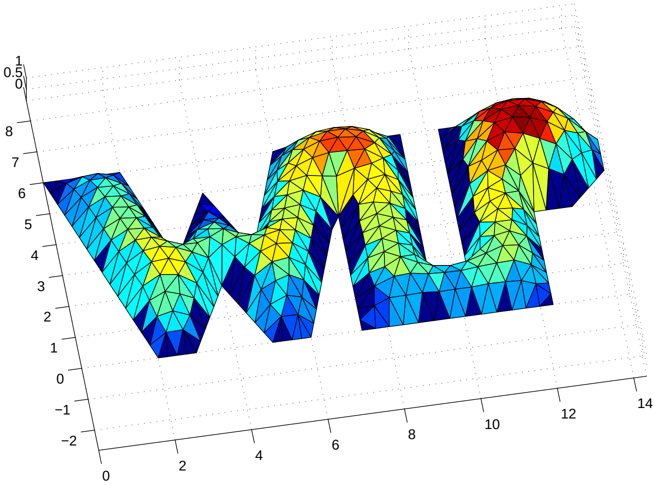
Example:

$$-\Delta u = \text{constant} \qquad \Omega$$
$$u = 0 \qquad \partial\Omega,$$

where Ω is given by



Example—FEM



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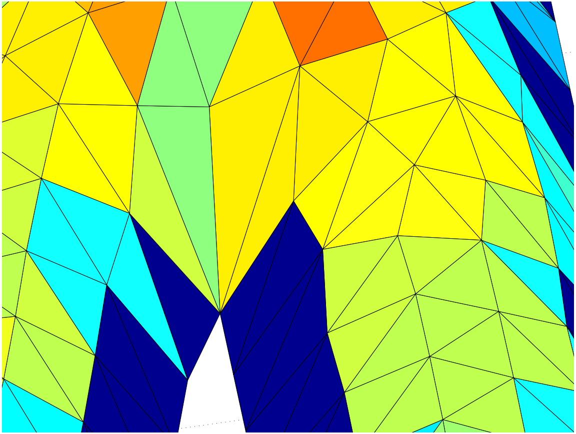

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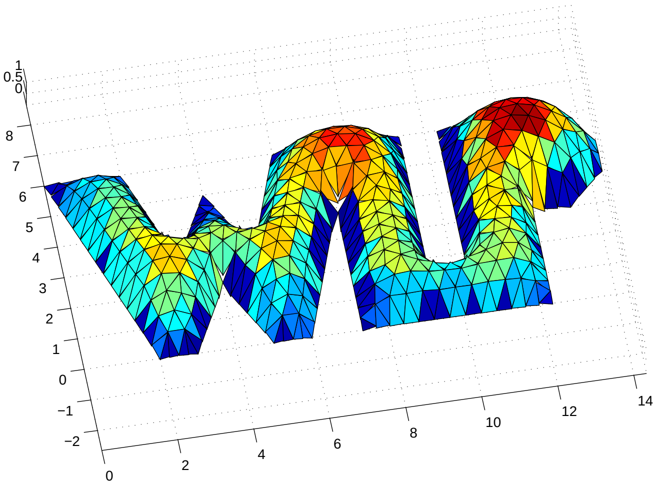

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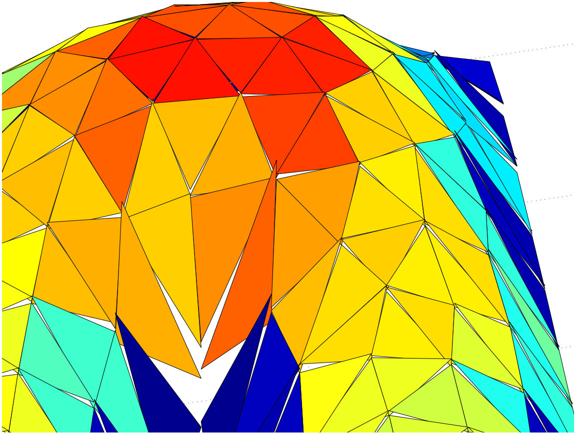
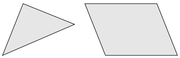
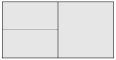
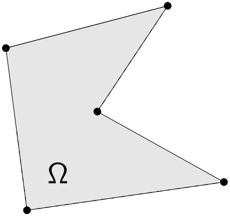


Why DGFEM?

- Great flexibility with respect to mesh design:
  - Different elements (shape, order): 
  - Irregular meshes: 
- Different kinds of (non-homogeneous) boundary conditions.
- Stability and robustness properties.
- Discontinuous data.
- Applicable to a wide variety of problems.

A Priori Analysis

- Let Ω be a polygonal domain:



- Typical solution behavior of second-order elliptic problems:
  - high smoothness (analyticity) in the interior of Ω.
  - low regularity at the corners ($H^{2-\epsilon}, 0 \leq \epsilon < 1$).

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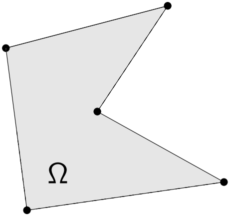

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A Priori Analysis

- Error analysis: Split the error $e_{DG} = u - u_{DG}$ into two parts,

$$e_{DG} = \underbrace{(u - \mathsf{I}_{DG}u)}_{=\eta} + \underbrace{(\mathsf{I}_{DG}u - u_{DG})}_{=\xi}.$$

Then,

$$C_1\|\xi\|_{DG}^2 \leq a_{DG}(\xi,\xi) = a_{DG}(e_{DG}-\eta,\xi) = -a_{DG}(\eta,\xi).$$

Hence,

$$\|\xi\|_{DG} \leq C_1^{-1} \sup_{\xi \in V_{DG}} \frac{|a_{DG}(\eta,\xi)|}{\|\xi\|_{DG}}.$$

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- Error analysis (cont.): There holds

$$\sup_{\xi \in V_{DG}} \frac{|a_{DG}(\eta, \xi)|}{\|\xi\|_{DG}} \leq C p_{\max} |||\eta|||,$$

where

$$|||\eta|||^2 = \|\eta\|^2_{H^1(\Omega,\mathcal{T})} + (\text{weighted } H^2\text{-seminorms of } \eta).$$

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Exponential Convergence

- Goal of *hp*-FEM/DGFEM: Exponential convergence with respect to $N = \dim V_{DG}$.
- Idea:
  - At the corners: Choose exponentially small elements with low polynomial degrees.
  - Away from the corners: Exploit the analyticity of the solution by using high polynomial degrees on large elements.
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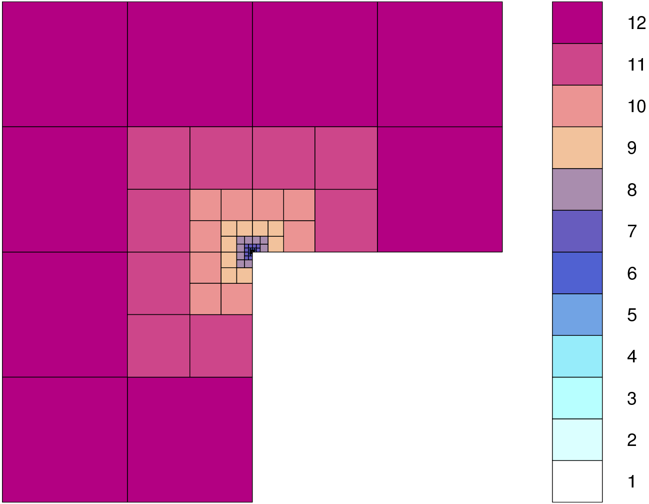


Exponential Convergence of *hp*-DGFEM

Theorem (Exponential Convergence)

There holds the a priori error estimate

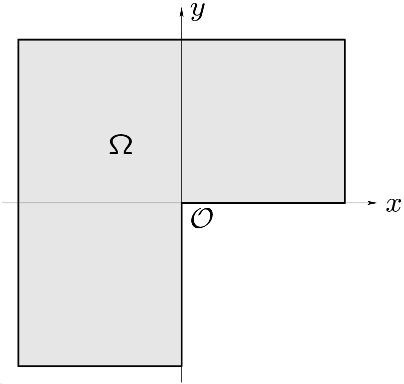
$$\|u - u_{DG}\|_{DG} \leq Ce^{-b\sqrt[3]{N}},$$

where the constants $C, b > 0$ *are independent of the element sizes and the polynomial degrees.*

P. Frauenfelder, C. Schwab, and T. W.
Comput. Math. Appl., 46:183–205, 2003.

Numerical Results

Model problem with re-entrant corner:



Exact solution:

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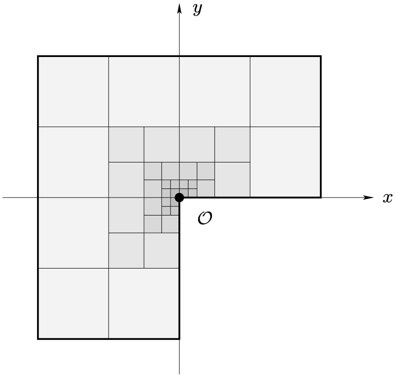

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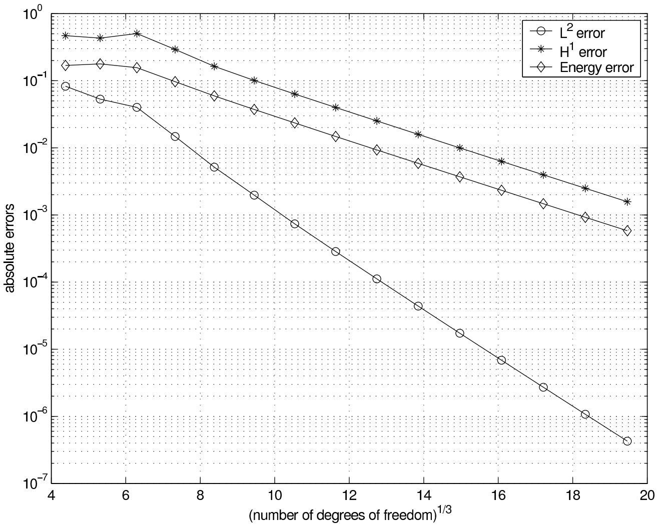

Part II

A Posteriori Error Analysis

Adaptivity

- Residual-based error estimation:

$$\|u - u_{DG}\|_{DG}^2 \le \sum_{K \in \mathcal{T}} \Phi_K(u_{DG}).$$

- FEM vs. DGFEM:

$$e_{FEM} = u - u_{FEM}, \qquad e_{DG} = u - u_{DG}.$$

FEM		DGFEM
$V_{FEM} \subset V$	$\longleftrightarrow$	$V_{DG} \not\subset V$
$a \equiv a_{FEM}$	$\longleftrightarrow$	$a \not\equiv a_{DG}$
$\|e_{FEM}\|^2 \lesssim a_{FEM}(e_{FEM}, e_{FEM})$ (coercivity, inf-sup cond.)	$\longleftrightarrow$	? $e_{DG} \not\in V$

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A Posteriori Error Analysis

- Idea: Decompose the DG finite element space as

$$V_{DG} = V_{DG}^{\|} \oplus V_{DG}^{\perp},$$

with

$$V_{DG}^{\|} = H_0^1(\Omega) \cap V_{DG} \subset H_0^1(\Omega)$$
$$V_{DG}^{\perp} = \text{orthogonal complement of } V_{DG}^{\|} \text{ in } V_{DG}.$$

- Then, let

$$u_{DG} = \underbrace{u_{DG}^{\|}}_{\in V_{DG}^{\|}} + \underbrace{u_{DG}^{\perp}}_{\in V_{DG}^{\perp}},$$

and

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A Posteriori Error Analysis

- Error splitting:

$$\begin{aligned}
\|e_{DG}\|_{DG}^2 &= \|u - u_{DG}\|_{DG}^2 \\
&= \|\nabla_h e_{DG}\|_{L^2(\Omega)}^2 + \gamma \int_{\mathcal{E}} \mathtt{h}^{-1}\mathtt{p}^2 |[\![u - u_{DG}]\!]|^2 \,\mathrm{d}s \\
&= \int_\Omega \nabla_h e_{DG} \cdot \nabla_h e_{DG} \,\mathrm{d}\mathbf{x} + \gamma \int_{\mathcal{E}} \mathtt{h}^{-1}\mathtt{p}^2 |[\![u_{DG}]\!]|^2 \,\mathrm{d}s \\
&= \int_\Omega \nabla_h e_{DG} \cdot \nabla_h e_{DG}^{\|} \,\mathrm{d}\mathbf{x} - \int_\Omega \nabla_h e_{DG} \cdot \nabla_h u_{DG}^{\perp} \,\mathrm{d}\mathbf{x} \\
&\quad + \gamma \int_{\mathcal{E}} \mathtt{h}^{-1}\mathtt{p}^2 |[\![u_{DG}]\!]|^2 \,\mathrm{d}s \\
&= T_1 - T_2 + T_3.
\end{aligned}$$

- Prove that: $|T_1 - T_2 + T_3| \leq \|e_{DG}\|_{DG} \left(\sum_{K\in\mathcal{T}} \Phi_K^2\right)^{\frac{1}{2}}$.

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A Posteriori Error Analysis

- There holds:

$$|T_1| \leq C \|e_{DG}\|_{DG} \,(\text{computable } residual(u_{DG}, f, \mathtt{h}, \mathtt{p}, \gamma))$$

and

$$|T_2| \leq \left| \int_\Omega \nabla_h e_{DG} \cdot \nabla_h u_{DG}^\perp \,\mathrm{d}\mathbf{x} \right| \leq \|e_{DG}\|_{DG} \|u_{DG}^\perp\|_{DG}$$

- Norm equivalence on $V_{DG}^\perp$ (for conforming meshes):

Proposition

$$\int_{\mathcal{E}} \mathtt{h}^{-1} \mathtt{p}^2 |[\![\phi]\!]|^2 \,\mathrm{d}s \simeq \|\phi\|_{DG}^2 \qquad \forall \phi \in V_{DG}^\perp.$$

P. Houston, D. Schötzau, and T. W.
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- Norm equivalence on $V_{DG}^\perp$ (for conforming meshes):

Proposition

$$\int_{\mathcal{E}} \mathtt{h}^{-1}\mathtt{p}^2 |[\![\phi]\!]|^2 \, \mathrm{d}s \simeq \|\phi\|_{DG}^2 \qquad \forall \phi \in V_{DG}^\perp.$$

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M3AS.

A Posteriori Error Analysis

- Then,

$$\|u_{DG}^{\perp}\|_{DG}^{2} \simeq \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^{2}|[\![u_{DG}^{\perp}]\!]|^{2}\,\mathrm{d}s$$
$$= \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^{2}|[\![u_{DG}^{\perp} + u_{DG}^{\|}]\!]|^{2}\,\mathrm{d}s$$
$$= \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^{2}|[\![u_{DG}]\!]|^{2}\,\mathrm{d}s.$$

- Hence,

$$|T_2| \leq \|e_{DG}\|_{DG}\|u_{DG}^{\perp}\|_{DG}$$
$$\leq C\|e_{DG}\|_{DG}\left(\gamma\int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^{2}\,|[\![u_{DG}]\!]|^{2}\,\mathrm{d}s\right)^{\frac{1}{2}}$$

- Then,

$$\|u_{DG}^{\perp}\|_{DG}^{2} \simeq \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^{2}|[\![u_{DG}^{\perp}]\!]|^{2}\,\mathrm{d}s = \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^{2}|[\![u_{DG}^{\perp} + u_{DG}^{\parallel}]\!]|^{2}\,\mathrm{d}s = \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^{2}|[\![u_{DG}]\!]|^{2}\,\mathrm{d}s.$$

- Hence,

$$|T_2| \leq \|e_{DG}\|_{DG}\|u_{DG}^{\perp}\|_{DG} \leq C\|e_{DG}\|_{DG}\left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^{2}\,|[\![u_{DG}]\!]|^{2}\,\mathrm{d}s\right)^{\frac{1}{2}}$$

- Furthermore,

$$
\begin{aligned}
|T_3| &= \gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 |[\![u_{DG}]\!]|^2 \,\mathrm{d}s \\
&= \gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, [\![u_{DG}]\!] \cdot [\![u_{DG} - u]\!] \,\mathrm{d}s \\
&= \gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, [\![u_{DG}]\!] \cdot [\![-e_{DG}]\!] \,\mathrm{d}s \\
&\leq \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, |[\![e_{DG}]\!]|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, |[\![u_{DG}]\!]|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \|e_{DG}\|_{DG} \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, |[\![u_{DG}]\!]|^2 \,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}
$$

A Posteriori Error Analysis

- Furthermore,

$$
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&\leq \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2\, |[\![e_{DG}]\!]|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2\, |[\![u_{DG}]\!]|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \|e_{DG}\|_{DG} \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2\, |[\![u_{DG}]\!]|^2 \,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}
$$

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$$
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&\leq \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, |[\![e_{DG}]\!]|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, |[\![u_{DG}]\!]|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\leq \|e_{DG}\|_{DG} \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, |[\![u_{DG}]\!]|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} .
\end{aligned}
$$

A Posteriori Error Analysis

- Furthermore,

$$
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&\le \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, |[\![e_{DG}]\!]|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, |[\![u_{DG}]\!]|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\le \|e_{DG}\|_{DG} \left(\gamma \int_{\mathcal{E}} \mathrm{h}^{-1}\mathrm{p}^2 \, |[\![u_{DG}]\!]|^2 \, \mathrm{d}s \right)^{\frac{1}{2}} .
\end{aligned}
$$

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$$
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\end{aligned}
$$

Theorem (hp-IPDG for $-\Delta u = f$)

Let the exact solution $u \in H_0^1(\Omega)$. Then, there holds the hp-a posteriori error estimate:

$$\|u - u_{DG}\|_{DG} \leq C \left(\sum_{K \in \mathcal{T}} \Phi_K^2 \right)^{\frac{1}{2}}.$$

The local error indicators Φ_K, $K \in \mathcal{T}$, are given by

$$\Phi_K^2 = h_K^2 p_K^{-2} \|f + \Delta u_{DG}\|_{L^2(K)}^2 + h_K p_K^{-1} \|[\![\nabla u_{DG}]\!]\|_{L^2(\partial K \setminus \partial\Omega)}^2 + \gamma h_K^{-1} p_K^2 \|[\![u_{DG}]\!]\|_{L^2(\partial K)}^2.$$

$C > 0$ is independent of the parametrization parameters.

Remark

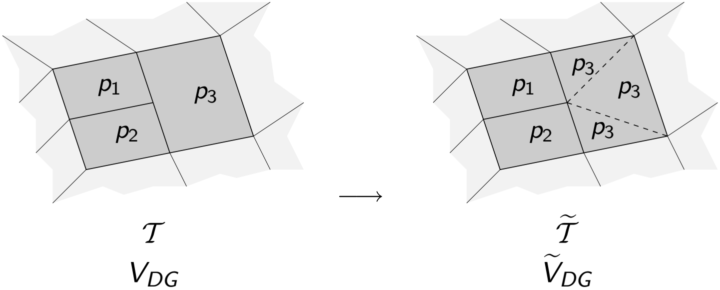
The proposed error estimator is efficient, i.e., local lower hp-error estimates can be proved.

This can be shown along the lines of

J. M. Melenk and B. I. Wohlmuth,
On residual-based a posteriori error estimation in *hp*-FEM
Adv. Comp. Math., 15:311-331, 2001.

Nonconforming meshes

- For the analysis with nonconforming meshes (containing hanging nodes) the DG space V_{DG} is "regularized":



Part III

Applications

Linearized Elasticty

- Given: Polygon $\Omega \subset \mathbb{R}^d$, external force $\mathbf{f} \in \mathbf{L}^2(\Omega)^d$, Lamé coefficients μ, λ for homogeneous isotropic materials.
- Problem: Find displacement $\mathbf{u} \in \mathbf{H}_0^1(\Omega)^d$ such that

$$-\nabla \cdot \underline{\sigma}(\mathbf{u}) = \mathbf{f} \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega,$$

where

$$\underline{\sigma}(\mathbf{u}) = 2\mu\underline{\varepsilon}(\mathbf{u}) + \lambda\, \nabla\cdot\mathbf{u}\, \mathbb{I}_{d\times d},$$

with

$$\underline{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top).$$

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with

$$\underline{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top).$$

- Stability:

$$\|\nabla u\|_{L^2(\Omega)} + \lambda\|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{L^2(\Omega)}.$$

- Incompressibility constraint:

$$\|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad \lambda \to \infty.$$

- For standard FEM: Volume Locking

- Stability:

$$\|\nabla u\|_{L^2(\Omega)} + \lambda \|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{L^2(\Omega)}.$$

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Stability / Volume Locking

- Stability:

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- For standard FEM: Volume Locking

Volume Locking: FEM vs. DGFEM ($d = 2$)

- Error estimate (linear elements, $d = 2$):

$$\|\mathbf{u} - \mathbf{u}_{FEM}\|_{Energy} \leq Ch.$$

- Standard FEM:

$$C = C(\lambda) \to \infty \text{ as } \lambda \to \infty.$$

- DGFEM, $\partial\Omega$ smooth:

$$C \text{ independent of } \lambda.$$

P. Hansbo & M. Larson, CMAME, 2001.

- DGFEM remains robust (free of volume locking) for non-smooth solutions.

T.W., IMA J. Numer. Anal., 2004.

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P. Hansbo & M. Larson, CMAME, 2001.

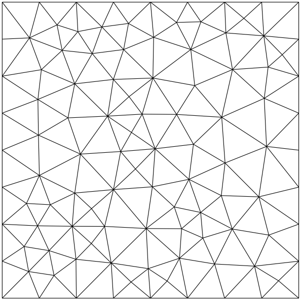
- DGFEM remains robust (free of volume locking) for non-smooth solutions.

T.W., IMA J. Numer. Anal., 2004.

- Model problem:

$$
\begin{aligned}
-\nabla \cdot \sigma(\mathbf{u}) &= \mathbf{0} \quad \text{in} \quad \Omega \\
\mathbf{u} &= \mathbf{g}_D \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

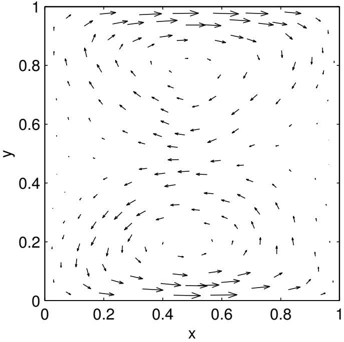
Exact solution:
$\mathbf{u} \in \mathbf{H}^2(\Omega)^2$

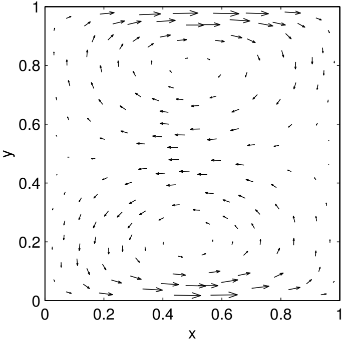


$\Omega = (0, 1)^2$

Volume Locking

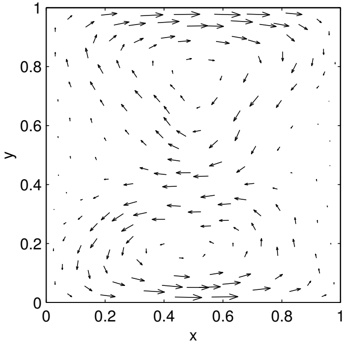
- FEM/DGFEM: $\lambda = 100$





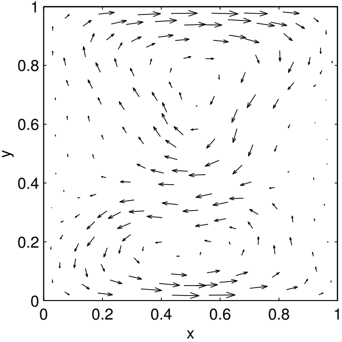
Volume Locking

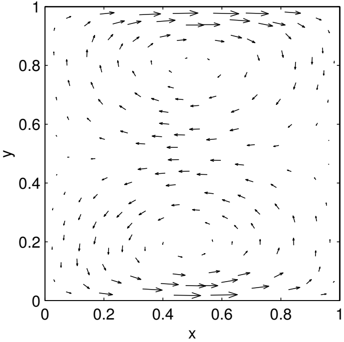
- FEM/DGFEM: $\lambda = 1000$



Volume Locking

- FEM/DGFEM: $\lambda = 5000$





- DG space:

$$\mathbf{V}_h = \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega)^d : \mathbf{v}|_K \in P_p(K)^d, K \in \mathcal{T} \right\}.$$

- Variational formulation: Find $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}) = l_h(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h.$$

- Forms:

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) = &\int_\Omega \underline{\sigma}_h(\mathbf{u}) : \underline{\varepsilon}_h(\mathbf{v})\,\mathrm{d}\mathbf{x} \\ &- \int_{\mathcal{E}} \{\!\{\underline{\sigma}_h(\mathbf{u})\}\!\} : \underline{\llbracket \mathbf{v} \rrbracket} + \underline{\llbracket \mathbf{u} \rrbracket} : \{\!\{\underline{\sigma}_h(\mathbf{v})\}\!\}\,\mathrm{d}s \\ &+ \gamma \int_{\mathcal{E}} \mathrm{h}^{-1} \underline{\llbracket \mathbf{u} \rrbracket} : \underline{\llbracket \mathbf{v} \rrbracket}\,\mathrm{d}s + \gamma\lambda^2 \int_{\mathcal{E}} \mathrm{h}^{-1} \llbracket \mathbf{u} \rrbracket \llbracket \mathbf{v} \rrbracket\,\mathrm{d}s. \end{aligned}$$

$$l_h(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}\mathbf{x}.$$

DGFEM

- DG space:

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- Forms:

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$$l_h(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}.$$

- DG space:

$$\mathbf{V}_h = \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega)^d : \mathbf{v}|_K \in P_p(K)^d, K \in \mathcal{T} \right\}.$$

- Variational formulation: Find $\mathbf{u}_h \in \mathbf{V}_h$ such that

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- Forms:

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) = &\int_\Omega \underline{\sigma}_h(\mathbf{u}) : \underline{\varepsilon}_h(\mathbf{v}) \, \mathrm{d}\mathbf{x} \\ &- \int_{\mathcal{E}} \{\!\{\underline{\sigma}_h(\mathbf{u})\}\!\} : \underline{[\![\mathbf{v}]\!]} + \underline{[\![\mathbf{u}]\!]} : \{\!\{\underline{\sigma}_h(\mathbf{v})\}\!\} \, \mathrm{d}s \\ &+ \gamma \int_{\mathcal{E}} \mathrm{h}^{-1} \underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]} \, \mathrm{d}s + \gamma\lambda^2 \int_{\mathcal{E}} \mathrm{h}^{-1} [\![\mathbf{u}]\!] [\![\mathbf{v}]\!] \, \mathrm{d}s. \end{aligned}$$

$$l_h(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}.$$

A Posteriori Error bound for Elasticity

Theorem ($\mathbf{H}^1$-norm)

Let $\mathbf{u} \in \mathbf{H}_0^1(\Omega)^d$ be the exact solution of the linear elasticity problem and $\mathbf{u}_h$ its DG approximation. Then, there holds the a posteriori error bound

$$\|\nabla_h(\mathbf{u}-\mathbf{u}_h)\|^2_{L^2(\Omega)^d} + \int_{\mathcal{E}} \mathtt{h}^{-1} \|\underline{[\![\mathbf{u}-\mathbf{u}_h]\!]}\|^2 \,\mathrm{d}s \leq C \sum_{K\in\mathcal{T}_h} \Phi_K^2,$$

with $C > 0$ independent of λ and of $\mathtt{h}$. The elemental error indicators Φ_K, $K \in \mathcal{T}_h$, are given by

$$\Phi_K^2 = h_K^2\|\mathbf{f}\|^2_{0,K} + h_K\|[\![\underline{\varepsilon}(\mathbf{u}_h)]\!]\|^2_{0,\partial K\setminus\partial\Omega} + \gamma^2 h_K^{-1}\|[\![\mathbf{u}_h]\!]\|^2_{0,\partial K}.$$

Furthermore, the error estimator is bounded independently of λ.

T.W., Math. Comp., 2006.

A Posteriori Error bound for Elasticity

Theorem (Energy-norm)

The previous error bound can be improved:

$$\|\nabla_h(\mathbf{u}-\mathbf{u}_h)\|^2_{L^2(\Omega)} + \lambda^2\|\nabla_h\cdot(\mathbf{u}-\mathbf{u}_h)\|^2_{L^2(\Omega)} + \int_{\mathcal{E}} \mathrm{h}^{-1}\|\underline{[\![\mathbf{u}-\mathbf{u}_h]\!]}\|^2\,\mathrm{d}s + \lambda^2\int_{\mathcal{E}} \mathrm{h}^{-1}\|[\![\mathbf{u}-\mathbf{u}_h]\!]\|^2\,\mathrm{d}s \leq C\sum_{K\in\mathcal{T}_h}\widetilde{\Phi}^2_K,$$

Furthermore, there hold corresponding (local) robust *lower bounds, i.e., the proposed error estimator is efficient.*

Remark: Analysis requires suitable inf-sup conditions.

Model problem:

Elasticity problem on a domain with re-entrant corner.

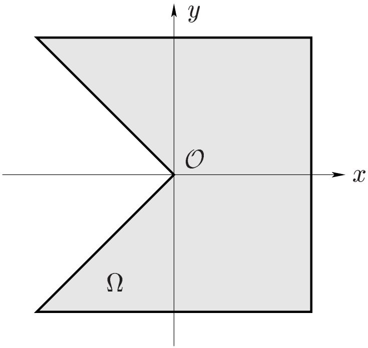
Exact solution:

$$\mathbf{u} \sim r^s \notin H^2(\Omega),$$

with

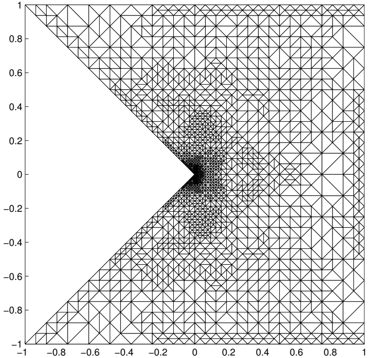
$$r(\mathbf{x}) = |\mathbf{x} - \mathcal{O}|,$$

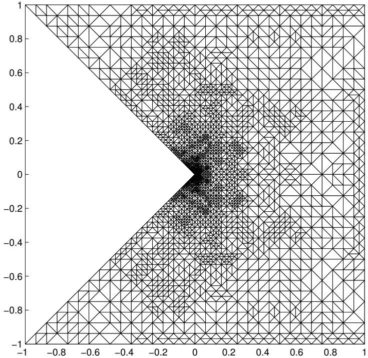
and $s = 0.54448\ldots$



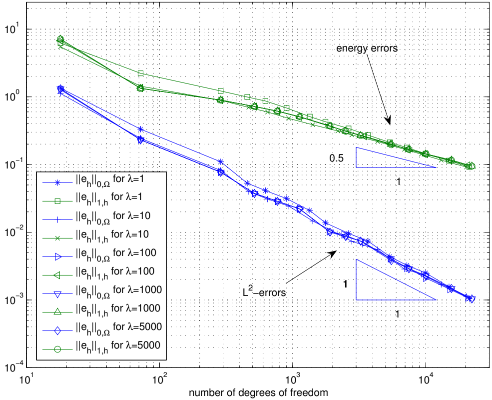
Numerical Results

Meshes for $\lambda = 1$ and $\lambda = 5000$ after 14 refinement steps:





Errors for the adaptive DGFEM:



Nonlinear Elliptic PDE

- Consider a (monotonic) quasilinear elliptic PDE:

$$\begin{aligned} -\nabla \cdot (\mu(\mathbf{x}, |\nabla u|)\nabla u) &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

- Non-linearity μ:

(A1) $\mu \in \mathcal{C}(\overline{\Omega} \times [0, \infty))$;

(A2) there exist positive constants m_μ and M_μ such hat

$$m_\mu(t - s) \leq \mu(\mathbf{x}, t)t - \mu(\mathbf{x}, s)s \leq M_\mu(t - s)$$

for all $t \geq s \geq 0$ and $\mathbf{x} \in \overline{\Omega}$.

Nonlinear Elliptic PDE

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$$\|u - u_{DG}\|_{DG}^2 \leq \sum_{K \in \mathcal{T}} \eta_K^2.$$

The local error indicators η_K, $K \in \mathcal{T}$, are given by

$$\begin{aligned} \eta_K^2 = {} & h_K^2 p_K^{-2} \|f + \nabla \cdot (\mu(|\nabla u_{DG}|)\nabla u_{DG})\|_{L^2(K)}^2 \\ & + h_K p_K^{-1} \|[\![\mu(|\nabla u_{DG}|)\nabla u_{DG}]\!]\|_{L^2(\partial K \setminus \partial\Omega)}^2 \\ & + \gamma^2 h_K^{-1} p_K^3 \|[\![u_{DG}]\!]\|_{L^2(\partial K)}^2. \end{aligned}$$

$C > 0$ is independent of the parametrization parameters.

P. Houston, E. Süli, and T. W.
To appear in *IMA J. Numer. Anal.*

hp-Adaptivity

- Goal: Exponential convergence.

h- or *p*-refinement ?

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P. Houston and E. Süli
A Note on the Design of hp-Adaptive Finite Element Methods for Elliptic Partial Differential Equations.
CMAME, 194:229–243, 2005.

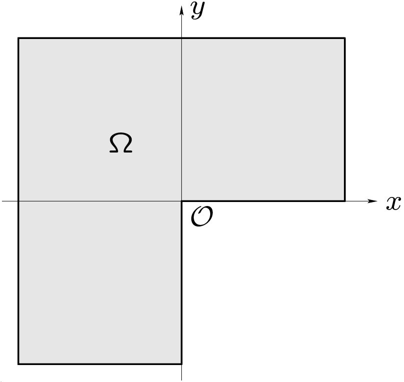
T. Eibner and M. Melenk
An adaptive strategy for hp-FEM based on testing for analyticity.
Comp. Mech., 2007.

Model problems with re-entrant corner:

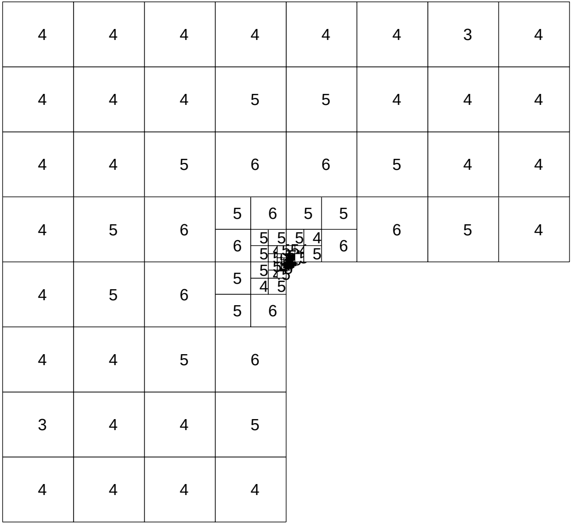
$$\mu(|\nabla u|) = 1 + e^{-|\nabla u|^2}$$

Exact solution:

$$u(r, \phi) = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\phi\right) \notin H^2(\Omega).$$



Numerical Results



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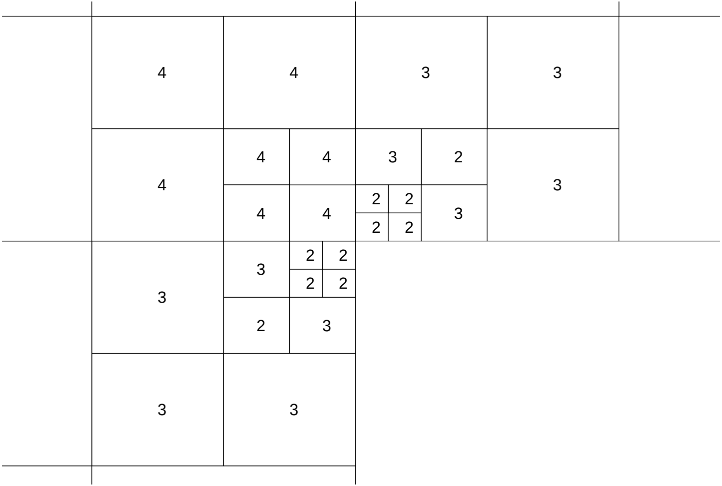

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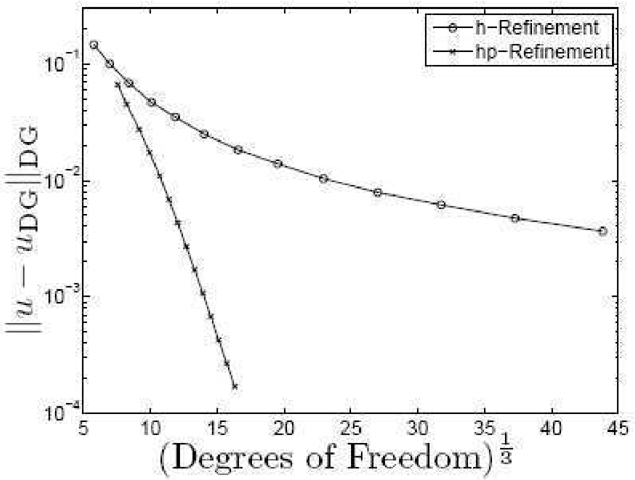

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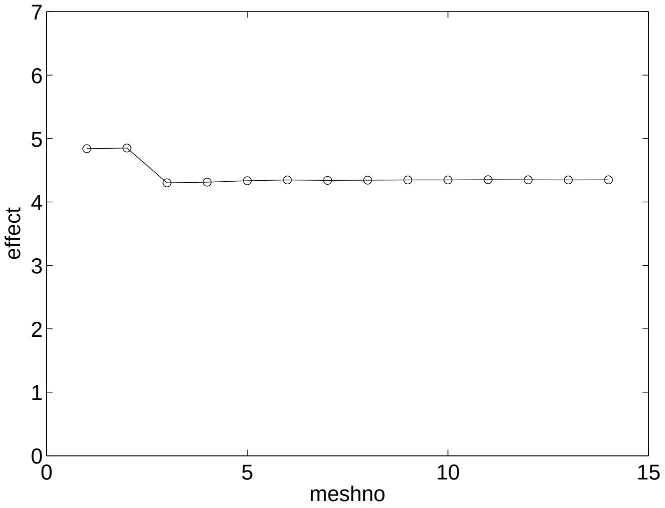

Summary & Outlook

- Error indicators for DGFEM, and *h*- and *hp*-adaptivity. General approach, applicable to other methods (nonforming, mixed, etc.).
- Applications: elasticity, Stokes, linear and quaslinear diffusion.
- *hp*-timestepping for parabolic PDE (cG/dG).
- Future work: 3-D, systems of nonlinear PDE (e.g., quasi-Newtonian flow), time-space adaptivity.