On Lagrangian Single-Particle Statistics

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Particle Tracking

- Seed flow with tracer particles

- Follow tracers in 3d in space and time

⇒ particle tracks
Why is it so difficult?

<table>
<thead>
<tr>
<th>Apparatus</th>
<th>$P$ (bar)</th>
<th>$\nu$ (m$^2$/s)</th>
<th>$u'$ (m/s)</th>
<th>$\epsilon$ (m$^2$/s$^3$)</th>
<th>$\ell$ (m)</th>
<th>$\lambda$ (µm)</th>
<th>$\eta$ (µm)</th>
<th>$\tau_\eta$ (ms)</th>
<th>$R_\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SF$_6$ tunnel</td>
<td>15</td>
<td>$1.5 \times 10^{-7}$</td>
<td>1.0</td>
<td>1.2</td>
<td>0.45</td>
<td>1400</td>
<td>7.3</td>
<td>0.36</td>
<td>9600</td>
</tr>
<tr>
<td>air tunnel</td>
<td>1</td>
<td>$1.5 \times 10^{-5}$</td>
<td>1.2</td>
<td>3.9</td>
<td>0.4</td>
<td>9100</td>
<td>172</td>
<td>2.0</td>
<td>730</td>
</tr>
<tr>
<td>SF$_6$ tank</td>
<td>15</td>
<td>$1.5 \times 10^{-7}$</td>
<td>1.0</td>
<td>5.5</td>
<td>0.094</td>
<td>648</td>
<td>5.0</td>
<td>0.17</td>
<td>4360</td>
</tr>
<tr>
<td>water tank</td>
<td>1</td>
<td>$8 \times 10^{-7}$</td>
<td>2.2</td>
<td>59</td>
<td>0.094</td>
<td>1000</td>
<td>9.7</td>
<td>0.12</td>
<td>2800</td>
</tr>
</tbody>
</table>
typical in 3d:

6000 fps
70000 fps

20000 particles
300 particles
• closed container
• no meanflow in middle
• driven by 1kW DC motors
• temperature controlled to 50mK
• water filtered to 0.3 microns

\[ \text{Re}_\lambda = 1000 \]
\[ (\text{Re} = 70,000) \]
- closed container
- no meanflow in middle
- driven by 1kW DC motors
- temperature controlled to 50mK
- water filtered to 0.3 microns

\[ \text{Re}_\lambda = 1000 \]
\[ (\text{Re} = 70.000) \]
Acceleration is extremely intermittent

\( R_\lambda = 690 \)
\( d = 25 \ \mu m \)
• wind speed 18km/h (5m/sec)
• height above ground 1m
• roughness height 0.05m
  (farmland with few trees in summer time)

\[ \tau_\eta = 5 \text{ msec and } \eta = 0.5 \text{ mm} \]

every 15 sec > 15g acceleration
Lagrangian Velocity Statistics

G. Falkovich, H. Xu, A. Pumir, EB, L. Biferale, G. Boffetta, A. Lanotte, F. Toschi
(to appear in Phys. Fluids)
**Eulerian**

Exact flux law (Kolmogorov Eq.)

\[
\langle (\delta^L_r u)^3 \rangle = -\frac{12}{d(d+2)} \epsilon r
\]

Eulerian velocity increments (K41 scaling)

\[
\delta_r u \sim (\epsilon r)^{1/3}
\]

**Lagrangian**

Lagrangian velocity increments (dimensional argument)

\[
\delta_\tau u \sim (\epsilon \tau)^{1/2}
\]

Assumption:

\[
\langle (\delta_\tau u)^2 \rangle \overset{?}{=} C_0 \epsilon \tau
\]
Sign of the flux

$$\epsilon > 0 \quad \text{for} \quad d=3$$

$$\epsilon < 0 \quad \text{for} \quad d=2$$

But \( D_2(\tau) \equiv \langle (\delta_\tau u)^2 \rangle \geq 0 \)

Moreover, if reverse time \( t \rightarrow -t \)

The sign of \( D_2(\tau) \) remain unchanged!
No scaling range observed from currently available DNS and experimental data.
$D_2(\tau) \equiv \langle (\delta_\tau u)^2 \rangle \approx \epsilon \tau$

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Sawford & Yeung (Phys. Fluids, 2011)
Acceleration auto-correlation

\[
\frac{d}{d\tau} \langle (\delta_\tau u)^2 \rangle = 2 \langle a(\tau)[u(\tau) - u(0)] \rangle \\
= 2 \int_0^{\tau} \langle a(0)a(t) \rangle dt
\]

\((\delta_\tau u)^2 \sim \tau \Rightarrow \) acceleration is uncorrelated over time-lag \(\tau\)

Kinematic constraint (Tennekes & Lumley (1972)):

\[
\int_0^\infty \langle a(0)a(t) \rangle dt = 0
\]

This restrict the shape of the acceleration auto-correlation.
Data from particle tracking measurements

Mordant, Crawford, Bodenschatz (PRL, 2004)
Data from DNS

zero-crossing at $\approx 2\tau_\eta$

Yeung et al. (JFM, 2007)
Zero-crossing of the acceleration auto-correlation gives the peak of

\[
\frac{d \langle (\delta_\tau u)^2 \rangle}{d\tau} = 2 \langle a \delta_\tau u \rangle
\]

\[
\langle (\delta_\tau u)^2 \rangle = C_0 \varepsilon \tau \quad \Rightarrow \quad 2 \langle a \delta_\tau u \rangle = C_0 \varepsilon
\]

However, experimental and DNS data suggest an exponential decay after the peak.
Acceleration spectra

Acceleration spectrum may show a wider scaling range than that of the velocity structure function (Lien & D’Asaro (Phys. Fluids, 2002), Sawford & Yeung (Phys. Fluids, 2011)).

\[ \langle (\delta_{\tau}u)^2 \rangle \sim \tau \quad \Rightarrow \quad \Phi_A(\omega) \sim \omega^0, \quad (1/T_L \lesssim \omega \lesssim 1/\tau_\eta) \]

Remark: A flat acceleration spectrum implies $\delta$-correlated acceleration.
DNS results from Sawford & Yeung (Phys. Fluids, 2011)
DNS results from Sawford & Yeung (Phys. Fluids, 2011)
Acceleration spectra suggest anomalous scaling for velocity increments:

\[ \Phi_A(\omega) \sim \omega^{\mu} \quad \Rightarrow \quad \langle (\delta_T u)^2 \rangle \sim t^{1-\mu} \]

Moreover, acceleration variance is:

\[ \langle a^2 \rangle = \int_0^\infty d\omega \Phi_A(\omega) \approx \int_1^{1/\tau_\eta} A_0 \epsilon \omega^{\mu} d\omega \]

\[ \sim \epsilon \tau_\eta^{-(1+\mu)} \sim \frac{\epsilon^{3/2}}{\nu^{1/2}} R_\lambda^{\mu} \]

Which implies:

\[ a_0 \equiv \frac{\langle a^2 \rangle \nu^{1/2}}{\epsilon^{3/2}} \sim R_\lambda^{\mu} \]

Consistent with observations from experiments and DNS.
Summary for part 1

Dimensional scaling for Lagrangian velocity structure functions is not consistent with either theoretical considerations or experimental/numerical data.

Using extended-self-similarity to the study of Lagrangian velocity structure functions is questionable.

Interpolation schemes that bridges viscous, inertial, and large scales can only be used with caution, as the scaling relations are built-in while constructing the scheme.

Future work is on the study of the relation between Lagrangian statistics and energy flux, to which multi-point, multi-time statistics is of great interest.
Path Length Statistics

N. T. Ouellette, EB, H. Xu

Displacement:
\[ \mathbf{R}(t) \equiv \mathbf{X}(t) - \mathbf{X}(0) \]

Taylor (1922):
\[ R(t) = |\mathbf{R}(t)| = \begin{cases} u'^2 t^2 & \text{if } t \ll T_L \\ u'^2 T_L t & \text{if } t \gg T_L \end{cases} \]

What about path length?

\[ S(t) \equiv \int_0^t |\mathbf{u}(t')| \, dt' \]
\[ \langle S^2 \rangle \sim t^2 \]
\[ \langle S^2 \rangle - \langle R^2 \rangle \sim t^3 \]
Local slope

\[
\frac{d \log \langle S^2 - R^2 \rangle}{d \log t}
\]
Q: Which physics determines the difference $S^2 - R^2$? Vortical structures?

Can we understand the scaling of $<S^2 - R^2>$?
Dimensional argument:

$$\langle S^2 - R^2 \rangle \sim \epsilon t^3$$

What about other terms like:

$$\langle (S - R)^2 \rangle \quad \langle SR \rangle$$

Note:

$$\langle S^2 - R^2 \rangle = \langle (S - R)^2 \rangle + 2\langle SR \rangle - 2\langle R^2 \rangle$$
$$\langle SR \rangle \sim t^2 \quad \text{for} \quad t \ll T_L$$
\[ \langle (S - R)^2 \rangle \sim t^{3.7} \quad \text{for } t \ll T_L \]
Are these special features of turbulence or some generic kinematic relations that are widely applicable?

Test with two other flows: a synthetic ABC flow and a Lagrangian stochastic model for single particle trajectories in turbulence.
Steady Arnold-Beltrami-Childress (ABC) flow:

\[
\mathbf{u}(x, y, z) = (A \sin kz + C \cos ky)\mathbf{e}_x \\
+ (B \sin kx + A \cos kz)\mathbf{e}_y \\
+ (C \sin ky + B \cos kx)\mathbf{e}_z
\]

\[
T_{ABC} = \frac{2\pi/k}{(A^2 + B^2 + C^2)^{1/2}}
\]

No scaling range at short times.
Results from stochastic model (Sawford, Phys. Fluids, 1991):

\[ R_\lambda = 815 \]

Similar results as in experiments, but with smaller inertial range.
Summary for part 2

For Lagrangian trajectories, path length and displacement scale similarly at short times.

The difference between path length and the displacement has interesting power-law scaling in the inertial range, which might be related to the Lagrangian structures in the turbulent flow.
Vortex stretching in turbulence

Expect alignment of vorticity with the strongest eigenvalue of the rate of strain tensor.

\[ \frac{D\omega_i}{Dt} = S_{ij} \omega_j + \nu \nabla^2 \omega_i \]
Instantaneous vorticity aligns with the intermediate eigenvalue of rate of strain tensor

How does vorticity evolve in time when following the flow in response to the initial stretching?

what is the behavior for dissipative and inertial scales?
perceived velocity gradients scale $r_0$
perceived velocity gradients scale $r_0$

\[ M : M \rho_i = \vec{u}_i - \frac{1}{4} \sum_i \vec{u}_i \]
perceived rate of strain and vorticity at scale $r_0$

perceived rate of strain:

$$S = \frac{1}{2} (M + M^T)$$

perceived vorticity:

$$\Omega = \frac{1}{2} (M - M^T)$$

Spectral code, up to $384^3$

$R_\lambda$ up to 170, $L_{\text{int}}/\eta \approx 300$

Seed particles that form initially isotropic tetrahedra and follow their motion

Particle tracking ~ 100 particles.

Isotropic tetrads:
4 particles within $(1 \pm 0.1)r_0$.
Vorticity aligns with **intermediate** eigenvalue of the strain
alignment dynamics
In the coordinates formed by the eigenvectors of strain \( \hat{e}_i(0) \)

\[
t_0 \equiv (r_0^2/\epsilon)^{1/3}
\]
alignment dynamics

Graphs showing the evolution of alignment dynamics over time. The graphs compare DNS (direct numerical simulation) and EXP (experimental) data for different conditions.
alignment dynamics

increase in vorticity and decrease in moment of inertia.
Vorticity dynamics:

\[ \frac{d\omega_i}{dt} = (1 - \alpha)S_{ij}\omega_j + \zeta\omega_i \]

Euler equation: \( \alpha = 0 \) and \( \zeta = 0 \)

Tetrad model: \( 0 < \alpha < 1 \)

In the frame of \( \hat{e}_i(0) \), the eigenvectors of \( S_{ij} \)

\[ \frac{d\omega_i}{dt} \bigg|_{t=0} = (1 - \alpha)\lambda_i\omega_i + \zeta\omega_i \]

Let \( c_i(t) \equiv \hat{e}_\omega(t) \cdot \hat{e}_i(0) \)

Note \( \omega_i = \omega c_i \) \( \sum_i c_i^2 = 1 \)
\[ \frac{dc_i^2}{dt} \bigg|_{t=0} = 2(1 - \alpha)c_i^2 \left( \lambda_i - \sum_j \lambda_j c_j^2 \right) \]

In component form:

\[ \frac{dc_1^2}{dt} \bigg|_{t=0} = 2(1 - \alpha)c_1^2 \left[ c_2^2(\lambda_1 - \lambda_2) + c_3^2(\lambda_1 - \lambda_3) \right] \geq 0 \]

\[ \frac{dc_2^2}{dt} \bigg|_{t=0} = 2(1 - \alpha)c_2^2 \left[ c_1^2(\lambda_2 - \lambda_1) + c_3^2(\lambda_2 - \lambda_3) \right] \geq 0 \quad \approx 0 \]

\[ \frac{dc_3^2}{dt} \bigg|_{t=0} = 2(1 - \alpha)c_3^2 \left[ c_1^2(\lambda_3 - \lambda_1) + c_2^2(\lambda_3 - \lambda_2) \right] \leq 0 \]

Note that \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \)
Vorticity dynamics:

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Note \( \omega_i = \omega c_i \quad \sum_i c_i^2 = 1 \)
summary

- perceived velocity gradient based on tetrads (4 points) in inertial scales.
- instantaneous alignment between vorticity and the intermediate stretching direction – also in inertial scales.
- dynamic alignment between vorticity and the strongest initial stretching direction.
- observed for inertial and dissipation scales.
- time scale is given \( t_0 \equiv (r_0^2 / \varepsilon)^{1/3} \)
- angular momentum is conserved for 0.1 \( t_0(r_0, \varepsilon) \).

Xu, Pumir, Bodenschatz (Nature Physics 7:709-712, 2011)