Dynamics of Vorticity Near the Position of its Maximum Modulus

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7 May 2012
Motivation

- Extreme events in realistic fluids: fields such as vorticity become intense and localised in space and time
- Finite-time singularity problem in ideal fluids
- One would like to understand how vorticity behaves near its maximum
- Does the position of the peak vorticity move with the flow? NO
- How is the spatial structure of vorticity near the peak vorticity?
Outline

1. Definitions and warming up
   - 3D Navier-Stokes fluid equations
   - Vorticity modulus $|\omega|$
   - Constantin’s equation and position of maximum vorticity modulus

2. Evolution of position of maximum vorticity modulus

3. Evolution of length scales of vorticity isosurfaces
3D Navier-Stokes fluid equations

\begin{align}
\frac{Du}{Dt} &= -\nabla p + \nu \Delta u, \\
\nabla \cdot u &= 0,
\end{align}

where \( u \equiv u(x, t) \) is the velocity vector field (assumed smooth), \( x \in \mathbb{R}^3, \ t \in [0, T_\ast) \), and \( \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \cdot \nabla \) is the Lagrangian derivative.

Vorticity vector field \( \omega \equiv \nabla \times u \) satisfies:

\begin{align}
\frac{D\omega}{Dt} &= (\nabla u)^T \omega + \nu \Delta \omega,
\end{align}

where \((\nabla u)^T \omega)_j = \frac{\partial u_j}{\partial x_k} \omega_k\), \( j = 1, 2, 3 \), in Cartesian coordinates (Einstein convention over repeated indices).
3D Navier-Stokes fluid equations

3D Navier-Stokes

\[
\frac{Du}{Dt} = -\nabla p + \nu \Delta u, \quad (1)
\]
\[
\nabla \cdot u = 0, \quad (2)
\]

where \( u \equiv u(x, t) \) is the velocity vector field (assumed smooth), \( x \in \mathbb{R}^3 \), \( t \in [0, T_*] \), and \( \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \cdot \nabla \) is the Lagrangian derivative.

Vorticity vector field \( \omega \equiv \nabla \times u \) satisfies:

\[
\frac{D\omega}{Dt} = (\nabla u)^T \omega + \nu \Delta \omega, \quad (3)
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where \( ((\nabla u)^T \omega)_j = \frac{\partial u_j}{\partial x_k} \omega_k \), \( j = 1, 2, 3 \), in Cartesian coordinates (Einstein convention over repeated indices).
Vorticity modulus $|\omega|$ (1/3)

\[
\frac{D\omega}{Dt} = (\nabla u)^T \omega + \nu \Delta \omega \quad \text{(Vorticity Equation)}
\]

Vorticity decomposition into modulus and direction:

\[
\omega = \omega \xi, \quad \omega \equiv |\omega|, \quad |\xi| \equiv 1.
\]

- Take the vorticity equation and evaluate the scalar product of each term with the vorticity vector field $\omega$. We get:

\[
\omega \cdot \frac{D\omega}{Dt} = \omega \frac{D\omega}{Dt} = \omega \cdot ((\nabla u)^T \omega + \nu \Delta \omega),
\]

\[
= \omega^2 \xi \cdot (\nabla u)\xi + \nu \omega \cdot \Delta \omega.
\]
Definitions and warming up

Evolution of position of maximum vorticity modulus

Evolution of length scales of vorticity isosurfaces

3D Navier-Stokes fluid equations

Vorticity modulus $|\omega|$ (1/3)

$$\frac{D\omega}{Dt} = (\nabla u)^T \omega + \nu \Delta \omega \quad \text{(Vorticity Equation)}$$

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$$= \omega^2 \xi \cdot (\nabla u) \xi + \nu \omega \cdot \Delta \omega.$$
Vorticity modulus $|\omega|$ (1/3)

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\frac{D\omega}{Dt} = (\nabla u)^T \omega + \nu \Delta \omega \quad \text{(Vorticity Equation)}
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\]

\[
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\]
Vorticity modulus $|\omega|$ (2/3)

\[
\frac{D\omega}{Dt} = \omega^2 \xi \cdot (\nabla u) \xi + \nu \omega \cdot \Delta \omega
\]

- A simple calculation yields

\[
\omega \cdot \Delta \omega = -\omega^2 |\nabla \xi|^2 + \omega \Delta \omega,
\]

so we get

\[
\frac{D\omega}{Dt} = \omega \xi \cdot (\nabla u) \xi + \nu \Delta \omega - \nu \omega |\nabla \xi|^2.
\]
Vorticity modulus $|\omega|$ (2/3)

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so we get

\[
\frac{D\omega}{Dt} = \omega \xi \cdot (\nabla u) \xi + \nu \Delta \omega - \nu \omega |\nabla \xi|^2.
\]
Now, defining the effective stretching rate $\alpha$ as:

$$\alpha \equiv \xi \cdot (\nabla u)\xi + \nu \Delta \omega - \nu \omega |\nabla \xi|^2,$$

we arrive at the Constantin-type evolution equation for the vorticity modulus:

$$\frac{D\omega}{Dt} = \omega \alpha.$$
Vorticity modulus $|\omega|$ (3/3)

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$$\frac{D\omega}{Dt} = \omega \alpha.$$
Constantin’s equation and position of maximum vorticity modulus (1/2)

Constantin’s equation (explicit form)

\[
\frac{\partial \omega}{\partial t}(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \omega(\mathbf{x}, t) = \omega(\mathbf{x}, t) \alpha(\mathbf{x}, t), \quad \forall \mathbf{x} \in \mathbb{R}^3, \quad \forall t \in [0, T^*]
\]

- Define the position of a local maximum of vorticity modulus \(\omega(\mathbf{x}, t)\) as the time-dependent vector \(\mathbf{Y}(t)\) such that:

\[
\nabla \omega(\mathbf{Y}(t), t) = 0, \quad \text{with} \quad \frac{\partial^2 \omega}{\partial x_j \partial x_k}(\mathbf{Y}(t), t) \quad \text{negative-definite}.
\]

- Evaluate Constantin’s equation at \(\mathbf{x} = \mathbf{Y}(t)\). The gradient term \(\nabla \omega(\mathbf{Y}(t), t)\) vanishes by definition and we get

\[
\frac{\partial \omega}{\partial t}(\mathbf{Y}(t), t) = \omega(\mathbf{Y}(t), t) \alpha(\mathbf{Y}(t), t), \quad \forall t \in [0, T^*).
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Constantin’s equation and position of maximum vorticity modulus (1/2)

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### Constantin’s equation and position of maximum vorticity modulus (1/2)

#### Constantin’s equation (explicit form)

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\]

- Define the position of a local maximum of vorticity modulus \(\omega(x, t)\) as the time-dependent vector \(Y(t)\) such that:
  \[
  \nabla \omega(Y(t), t) = 0, \quad \text{with} \quad \frac{\partial^2 \omega}{\partial x_j \partial x_k}(Y(t), t) \quad \text{negative-definite}.
  \]

- Evaluate Constantin’s equation at \(x = Y(t)\). The gradient term \(\nabla \omega(Y(t), t)\) vanishes by definition and we get
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  \frac{\partial \omega}{\partial t}(Y(t), t) = \omega(Y(t), t) \alpha(Y(t), t), \quad \forall t \in [0, T_*).
  \]
Constantin’s equation and position of maximum vorticity modulus (2/2)

\[ \frac{\partial \omega}{\partial t} (\boldsymbol{Y}(t), t) = \omega(\boldsymbol{Y}(t), t) \alpha(\boldsymbol{Y}(t), t), \quad \forall \, t \in [0, T_*) \]

- Notice now that

\[ \frac{d}{dt} [\omega(\boldsymbol{Y}(t), t)] = \frac{\partial \omega}{\partial t} (\boldsymbol{Y}(t), t) + \frac{d\boldsymbol{Y}}{dt} \cdot \nabla \omega(\boldsymbol{Y}(t), t) = \frac{\partial \omega}{\partial t} (\boldsymbol{Y}(t), t). \]

Comparing this with the boxed equation gives finally:

\[ \frac{d}{dt} [\omega(\boldsymbol{Y}(t), t)] = \omega(\boldsymbol{Y}(t), t) \alpha(\boldsymbol{Y}(t), t), \quad \forall \, t \in [0, T_*) . \]

- Up to here, it is not obvious whether or not \( \boldsymbol{Y}(t) \) follows the material particles (but it doesn’t).
Definitions and warming up
Evolution of position of maximum vorticity modulus
Evolution of length scales of vorticity isosurfaces

Constantin’s equation and position of maximum vorticity modulus (2/2)

\[ \frac{\partial \omega}{\partial t}(\mathbf{Y}(t), t) = \omega(\mathbf{Y}(t), t) \alpha(\mathbf{Y}(t), t), \quad \forall t \in [0, T^*_\ast) \]

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Constantin’s equation and position of maximum vorticity modulus (2/2)

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\frac{\partial \omega}{\partial t} (Y(t), t) = \omega(Y(t), t) \alpha(Y(t), t), \quad \forall \, t \in [0, T_*)
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Notice now that

\[
\frac{d}{dt} [\omega(Y(t), t)] = \frac{\partial \omega}{\partial t} (Y(t), t) + \frac{dY}{dt} \cdot \nabla \omega(Y(t), t) = \frac{\partial \omega}{\partial t} (Y(t), t).
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Constantin’s equations: Test of numerical data (1/3)

\[ \frac{d}{dt} [\omega(Y(t), t)] = \omega(Y(t), t) \alpha(Y(t), t), \quad \forall t \in [0, T_\star) \]

Choose \( Y(t) \) to be the position of the global maximum of vorticity modulus, so \( \omega(Y(t), t) = \| \omega(\cdot, t) \|_\infty \) (max norm).

We investigate this max norm using data from a 1024 \times 256 \times 2048 pseudo-spectral numerical simulation of 3D Euler anti-parallel vortices (Bustamante&Kerr 2007).
Choose $Y(t)$ to be the position of the global maximum of vorticity modulus, so $\omega(Y(t), t) = \|\omega(\cdot, t)\|_\infty$ (max norm).

We investigate this max norm using data from a $1024 \times 256 \times 2048$ pseudo-spectral numerical simulation of 3D Euler anti-parallel vortices (Bustamante & Kerr 2007).
Constantin’s equations: Test of numerical data (2/3)

- The position $Y(t)$ is trapped on the “symmetry plane”.

- We have stored spatial field data at the symmetry plane, at selected times $t$ between 5.9 and 9.4.

- At each selected time $t$, a spline spatial interpolation is done to obtain accurate values of the position of vorticity maximum $Y(t)$. 
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Lagrange vs. Euler – WPI, Vienna, Austria, 7-10 May 2012
Definitions and warming up

Evolution of position of maximum vorticity modulus
Evolution of length scales of vorticity isosurfaces

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Definitions and warming up
Evolution of position of maximum vorticity modulus
Evolution of length scales of vorticity isosurfaces

3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$
Constantin's equation and position of maximum vorticity modulus

Vorticity contours near vort. max., at time $t = 5.9375$

Vorticity contours on half plane, at time $t = 5.9375$
Definitions and warming up
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Evolution of length scales of vorticity isosurfaces

3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$
Constantin’s equation and position of maximum vorticity modulus

Vorticity contours near vort. max., at time $t = 6.5625$

Vorticity contours on half plane, at time $t = 6.5625$
Definitions and warming up
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3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$
Constantin's equation and position of maximum vorticity modulus

Vorticity contours near vort. max., at time $t = 6.875$

Vorticity contours on half plane, at time $t = 6.875$
Definitions and warming up
Evolution of position of maximum vorticity modulus
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3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$ 
Constantin's equation and position of maximum vorticity modulus

Vorticity contours near vort. max., at time $t = 7.1875$

Vorticity contours on half plane, at time $t = 7.1875$
Definitions and warming up
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3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$ 

Constantin's equation and position of maximum vorticity modulus

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3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$
Constantin's equation and position of maximum vorticity modulus

Vorticity contours near vort. max., at time $t = 7.8125$

Vorticity contours on half plane, at time $t = 7.8125$
Definitions and warming up
Evolution of position of maximum vorticity modulus
Evolution of length scales of vorticity isosurfaces

3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$
Constantin's equation and position of maximum vorticity modulus

Vorticity contours near vort. max., at time $t = 8.125$

Vorticity contours on half plane, at time $t = 8.125$
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Evolution of length scales of vorticity isosurfaces

3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$
Constantin's equation and position of maximum vorticity modulus

Vorticity contours near vort. max., at time $t = 8.4375$

Vorticity contours on half plane, at time $t = 8.4375$
Definitions and warming up
Evolution of position of maximum vorticity modulus
Evolution of length scales of vorticity isosurfaces

3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$
Constantin's equation and position of maximum vorticity modulus

Vorticity contours near vort. max., at time $t = 8.75$

Vorticity contours on half plane, at time $t = 8.75$
Definitions and warming up
Evolution of position of maximum vorticity modulus
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3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$,
Constantin's equation and position of maximum vorticity modulus

Vorticity contours near vort. max., at time $t = 9.0625$

Vorticity contours on half plane, at time $t = 9.0625$
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Evolution of position of maximum vorticity modulus
Evolution of length scales of vorticity isosurfaces

3D Navier-Stokes fluid equations
Vorticity modulus $|\omega|$
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Definitions and warming up

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3D Navier-Stokes fluid equations

Vorticity modulus $|\omega|$}

Constantin’s equation and position of maximum vorticity modulus

Spline–interpolated max vort position $Y(t)$ at selected times

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Lagrange vs. Euler – WPI, Vienna, Austria, 7-10 May 2012
Constantin’s equations: Test of numerical data (3/3)

\[
\frac{d}{dt} [\omega(Y(t), t)] = \omega(Y(t), t) \alpha(Y(t), t), \quad \forall t \in [0, T_*)
\]

We test the data by evaluating independently the values of \(\omega(Y(t), t)\) (green and red bullets), and the time integral of the time-interpolated product \(\omega(Y(t), t)\alpha(Y(t), t)\) (blue curve).
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Outline

1. Definitions and warming up

2. Evolution of position of maximum vorticity modulus
   - Drift equation
   - Understanding the drift

3. Evolution of length scales of vorticity isosurfaces
Evolution of position of maximum vorticity modulus $Y(t)$ (1/2)

By definition: 

$$\frac{\partial \omega}{\partial x_j}(Y(t), t) = 0, \quad \forall \ t \in [0, T_*), \quad j = 1, 2, 3.$$ 

Take time derivative of the above equation. We get:

$$\frac{d}{dt} \left[ \frac{\partial \omega}{\partial x_j}(Y(t), t) \right] = 0 = \frac{\partial^2 \omega}{\partial t \partial x_j}(Y(t), t) + \frac{dY}{dt} \cdot \frac{\partial \nabla \omega}{\partial x_j}(Y(t), t).$$

The first term in the RHS of this equation can be simplified using Constantin’s equation. We have in general:

$$\frac{\partial^2 \omega}{\partial t \partial x_j}(x, t) = -u(x, t) \cdot \frac{\partial \nabla \omega}{\partial x_j}(x, t) - \frac{\partial u}{\partial x_j} \cdot \nabla \omega(x, t)$$

$$+ \frac{\partial \omega}{\partial x_j}(x, t) \alpha(x, t) + \omega(x, t) \frac{\partial \alpha}{\partial x_j}(x, t).$$
Evolution of position of maximum vorticity $Y(t)$ (1/2)

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Evolution of position of maximum vorticity $Y(t)$ (1/2)

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- The first term in the RHS of this equation can be simplified using Constantin’s equation. We have in general:

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$$+ \frac{\partial \omega}{\partial x_j}(x, t) \alpha(x, t) + \omega(x, t) \frac{\partial \alpha}{\partial x_j}(x, t).$$
Evolution of position of maximum vorticity $Y(t)$ (2/2)

Evaluating this at $x = Y(t)$ we conclude:

$$0 = \left[ \frac{dY}{dt} - \mathbf{u}(Y(t), t) \right] \cdot \frac{\partial \nabla \omega}{\partial x_j}(Y(t), t) + \omega(Y(t), t) \frac{\partial \alpha}{\partial x_j}(Y(t), t)$$

so, in terms of the matrix of 2nd derivatives (i.e., Hessian) of $\omega$,

$$D^2 \omega(x, t) \equiv \begin{bmatrix} \frac{\partial^2 \omega}{\partial x_j \partial x_k} \end{bmatrix}(x, t),$$

which is by definition negative-definite at $x = Y(t)$ and therefore invertible there, we get the "drift" equation:

$$\frac{dY}{dt} = \mathbf{u}(Y(t), t) + \omega(Y(t), t) \left[ -D^2 \omega(Y(t), t) \right]^{-1} \nabla \alpha(Y(t), t).$$
Evolution of position of maximum vorticity $\mathbf{Y}(t)$ (2/2)

Evaluating this at $\mathbf{x} = \mathbf{Y}(t)$ we conclude:

$$0 = \left[ \frac{d\mathbf{Y}}{dt} - \mathbf{u}(\mathbf{Y}(t), t) \right] \cdot \frac{\partial \nabla \omega}{\partial x_j}(\mathbf{Y}(t), t) + \omega(\mathbf{Y}(t), t) \frac{\partial \alpha}{\partial x_j}(\mathbf{Y}(t), t)$$

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$$\frac{d\mathbf{Y}}{dt} = \mathbf{u}(\mathbf{Y}(t), t) + \omega(\mathbf{Y}(t), t) \left[ -D^2 \omega(\mathbf{Y}(t), t) \right]^{-1} \nabla \alpha(\mathbf{Y}(t), t).$$
Evaluation of position of maximum vorticity $Y(t)$ (2/2)

Evaluating this at $x = Y(t)$ we conclude:

$$0 = \left[ \frac{dY}{dt} - u(Y(t), t) \right] \cdot \frac{\partial \nabla \omega}{\partial x_j} (Y(t), t) + \omega(Y(t), t) \frac{\partial \alpha}{\partial x_j} (Y(t), t)$$

so, in terms of the matrix of 2nd derivatives (i.e., Hessian) of $\omega$,

$$D^2 \omega(x, t) \equiv \left[ \frac{\partial^2 \omega}{\partial x_j \partial x_k} \right] (x, t),$$

which is by definition negative-definite at $x = Y(t)$ and therefore invertible there, we get the "drift" equation:

$$\frac{dY}{dt} = u(Y(t), t) + \omega(Y(t), t) \left[ -D^2 \omega(Y(t), t) \right]^{-1} \nabla \alpha(Y(t), t).$$
Drift equation

\[ \frac{dY}{dt} = u(Y(t), t) + \omega(Y(t), t) \left[ -D^2 \omega(Y(t), t) \right]^{-1} \nabla \alpha(Y(t), t). \]

So the position of the global maximum of vorticity does not follow the material particles.

We define the “drift vector field” \( \mathcal{D}(x, t) \) for \( x \) near \( Y(t) \):

\[ \mathcal{D}(x, t) \equiv \omega(x, t) \left[ -D^2 \omega(x, t) \right]^{-1} \nabla \alpha(x, t). \]

Therefore the Drift equation is simply

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\]

So the position of the global maximum of vorticity does not follow the material particles.

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\[
\mathfrak{D}(x, t) \equiv \omega(x, t) \left[-D^2 \omega(x, t)\right]^{-1} \nabla \alpha(x, t).
\]

Therefore the Drift equation is simply

\[
\frac{dY}{dt} = u(Y(t), t) + \mathfrak{D}(Y(t), t).
\]
Definitions and warming up

Evolution of position of maximum vorticity modulus

Evolution of length scales of vorticity isosurfaces

Drift equation
Understanding the drift

Drift equation: Test of numerical data: \( x \)-coordinate

\[
\frac{dY}{dt} = u(Y(t), t) + \mathfrak{D}(Y(t), t),
\]

\[
\mathfrak{D}(x, t) = \omega(x, t) \left[ -D^2 \omega(x, t) \right]^{-1} \nabla \alpha(x, t).
\]

\[ Y(t) \quad \& \quad Y(t_0) + \int_{t_0}^{t} \{ u(Y(s), s) + \mathfrak{D}(Y(s), s) \} ds \]

\[ Y(t) \quad \& \quad Y(t_0) + \int_{t_0}^{t} u(Y(s), s) ds \]
Drift equation: Test of numerical data: z-coordinate

\[ \frac{dY}{dt} = u(Y(t), t) + \mathcal{D}(Y(t), t), \]

\[ \mathcal{D}(x, t) = \omega(x, t) \left[ -D^2 \omega(x, t) \right]^{-1} \nabla \alpha(x, t). \]
Understanding the drift

$$\mathcal{D}(x, t) = \omega(x, t) \left[ -D^2 \omega(x, t) \right]^{-1} \nabla \alpha(x, t)$$

The drift vector points *more or less* in the direction of $\nabla \alpha(Y(t), t)$, but this depends on the local profile of vorticity modulus near the maximum. See $t = 5.9$ snapshot:
\[ \mathcal{D}(x, t) = \omega(x, t) \left[ -D^2 \omega(x, t) \right]^{-1} \nabla \alpha(x, t). \]

Key quantities: eigenvalues of \( \omega(Y(t), t) \left[ -D^2 \omega(Y(t), t) \right]^{-1} \). Their square roots define three independent length scales, \( \lambda_1(t), \lambda_2(t), \lambda_3(t) \). Interpretation: as radii of the “nominal” ellipsoids of half-peak vorticity isosurfaces.
Definitions and warming up
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Understanding the drift

\[ \mathcal{S}(x, t) = \omega(x, t) \left[ -D^2 \omega(x, t) \right]^{-1} \nabla \alpha(x, t). \]

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Outline

1. Definitions and warming up

2. Evolution of position of maximum vorticity modulus

3. Evolution of length scales of vorticity isosurfaces
   - Direct study from numerical data
   - Equations of motion for length scales
   - Application: vortex blob’s circulation
Direct computation of eigenvalues of matrix

$$\sqrt{\omega(Y(t), t) \left[ -D^2 \omega(Y(t), t) \right]^{-1}}$$

at each selected time, gives the following symmetry-plane length scales:
Equations of motion for length scales

Each of the three length scales satisfies an equation of motion. We state these without proof:

$$\frac{d\lambda_a}{dt} = \lambda_a \mathbf{v}_a \cdot \left[ (\nabla \mathbf{u}) + \frac{1}{2} (\nabla \mathbf{D}) \right] \mathbf{v}_a, \quad a = 1, 2, 3,$$

where $\mathbf{v}_a$ are the normalised eigenvectors of $[D^2\omega(Y(t), t)]$.

Application: it is possible to determine how much does the vorticity profile deviate from self-similarity. Self-similar collapse at the symmetry plane would imply that the “vortex blob” has constant circulation:

$$C(t) \equiv \lambda_{\text{small}}(t) \lambda_{\text{Large}}(t) \| \omega(\cdot, t) \|_\infty = \text{const.}$$

Instead, we have, rigorously:

$$\frac{d}{dt} \ln C(t) = \frac{1}{2} \nabla_{2D} \cdot \mathbf{D}(Y(t), t)$$
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\[
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\]

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Miguel D. Bustamante

Lagrange vs. Euler – WPI, Vienna, Austria, 7-10 May 2012
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Vortex blob’s circulation

\[
\frac{d}{dt} \ln C(t) = \frac{1}{2} \nabla_2 D \cdot \mathfrak{D}(Y(t), t)
\]

\[
C(t) \quad \& \quad C(t_0) e^{\frac{1}{2} \int_{t_0}^{t} \nabla_2 D \cdot \mathfrak{D}(Y(s), s) \, ds}
\]

Blob's Circulation

Miguel D. Bustamante
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Conclusions

- We have revealed the laws of motion of the position of the vorticity maximum in 3D Navier-Stokes and Euler
- Fundamental role of new “Drift” vector field
- These laws have been used to check validity of high-resolution numerical simulations
- Fundamental role of the length scales of the vorticity profile near the maximum
- Implications regarding collapse self-similarity
- Numerical application of length-scale evolution equations leads to discovery of small-scale errors
- Work in progress: Errors are eliminated by looking at the slightly mollified version of the underlying PDE (Navier-Stokes or Euler)
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Thank you for your attention!