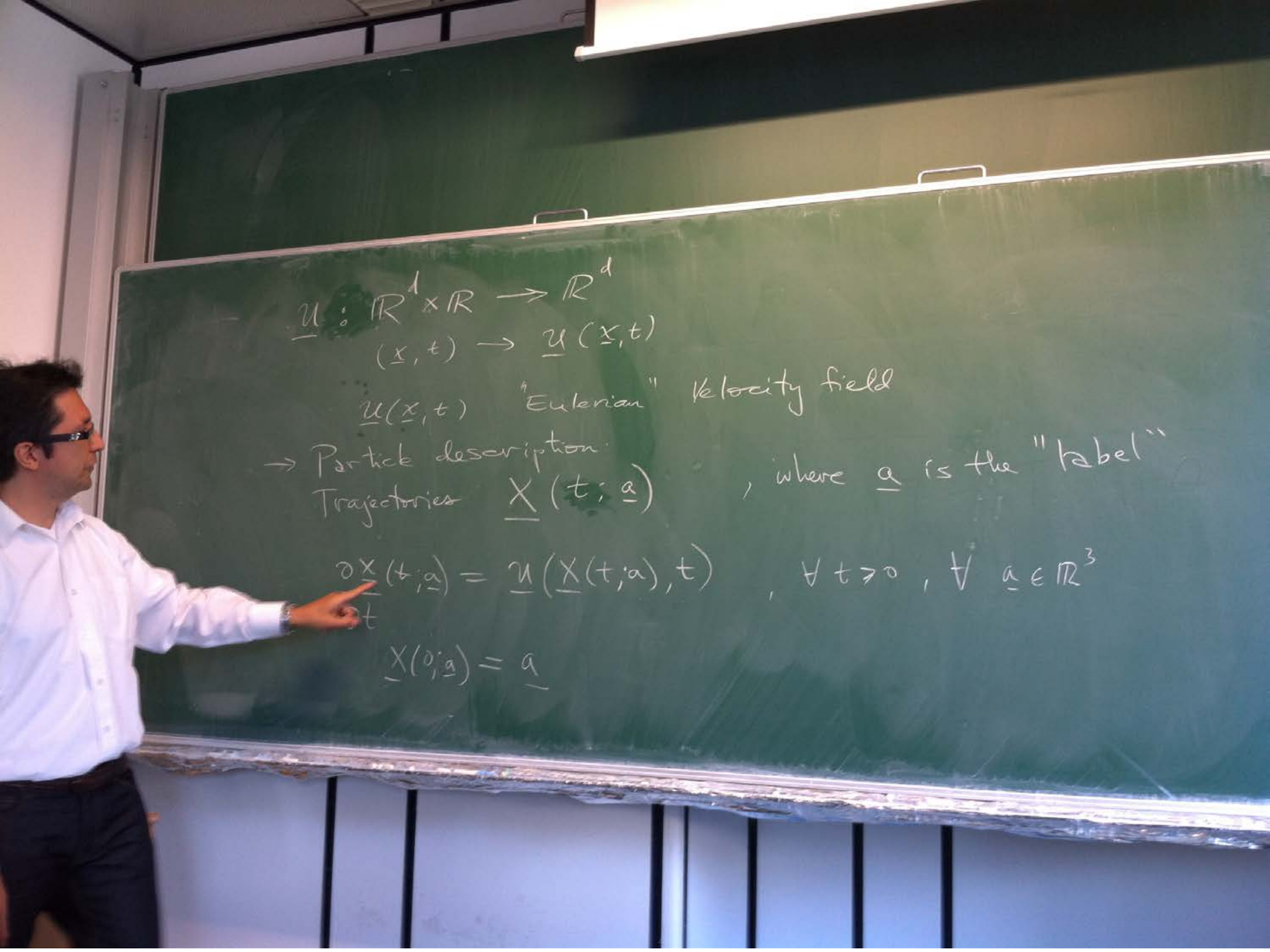

u : ℝ^d × ℝ → ℝ^d
(x, t) → u(x, t)
u(x, t) "Eulerian" velocity field
→ Particle description
Trajectories X(t; a) , where a is the "label"
∂X/∂t (t; a) = u(X(t; a), t) , ∀ t ≥ 0 , ∀ a ∈ ℝ^3
X(0; a) = a

A local variation of trajectories

$$\underline{X}(t;a) \longrightarrow \underline{X}(t;a) + \underbrace{\underline{\eta}(\underline{X}(t;a), t)}_{\text{"small" infinitesimally}}$$

will induce a ~~local~~ change in the velocity field

$$\underline{u}(\underline{x},t) \longrightarrow \underline{u}(\underline{x},t) + \underline{\delta u}(\underline{x},t) \qquad \forall\, t \geq 0,\ \forall\, \underline{x} \in \mathbb{R}^d$$

if we impose the condition

$$\frac{\partial}{\partial t}\left[\underline{X}(t;a) + \underline{\eta}(\underline{X}(t;a),t)\right] = u\left(\underline{X}(t;a) + \underline{\eta}(\underline{X}(t;a),t),t\right) + \delta u\left(\underline{X}(t;a) + \underline{\eta}(\underline{X}(t;a),t),t\right)$$

The idea is that $\underline{u}(\underline{x},t)$ is a funct

of $\underline{X}(t;a)$.

$$\frac{\partial \underline{X}(t;a)}{\partial t} = \underline{u}(\underline{X}(t;a)$$

A local variation of trajectories

$$\underline{X}(t;a) \longrightarrow \underline{X}(t;a) + \underbrace{\eta}_{\text{"small" } \text{infinitesimally}}(\underline{X}(t;\underline{a}), t)$$

will induce a ~~local~~ change in the velocity field

$$\underline{u}(\underline{x},t) \longrightarrow \underline{u}(\underline{x},t) + \underline{\delta u}(\underline{x},t) \qquad \forall\, t \geq 0,\ \forall\, \underline{x} \in \mathbb{R}^d$$

if we impose the condition

$$\frac{\partial}{\partial t}\left[\underline{X}(t;a) + \underline{\eta}(\underline{X}(t;a),t)\right] = u\Big(\underline{X}(t;\underline{a}) + \underline{\eta}(\underline{X}(t;\underline{a}),t), t\Big)$$

$$+ \delta u\Big(\underline{X}(t;a) + \eta\,\cancel{(\underline{X}(t;a),t)}, t\Big)$$

$$+ O(\eta^2)$$

The idea is

$\underline{u}(\underline{x},t)$ is

of $\underline{X}(t;a$

$$\frac{\partial \underline{X}(t;a)}{\partial t} = \underline{u}$$

$$\Rightarrow \cancel{\underline{u}(\underline{X}(t;\underline{a}),t)} + \underline{u}(\underline{X}(t;\underline{a}),t)\cdot\nabla\,\underline{\eta}(\underline{X}(t;\underline{a}),t) + \frac{\partial\underline{\eta}}{\partial t}(\underline{X}(t;\underline{a}),t)$$

$$= \cancel{\underline{u}(\underline{X}(t;\underline{a}),t)} + \underline{\eta}(\underline{X}(t;a),t)\cdot\nabla\,\underline{u}(\underline{X}(t;a),t)$$

$$+\ \underline{\delta u}(\underline{X}(t;a),t) \qquad + O(\eta^2)$$

$$\forall\, t \geq 0$$
$$\forall\, \underline{a} \in \mathbb{R}^d$$

$$\Rightarrow \boxed{\underline{\delta u}(\underline{x},t) = \frac{\partial\underline{\eta}}{\partial t}(\underline{x},t) + \underline{u}\cdot\nabla\,\underline{\eta}(\underline{x},t) - \underline{\eta}\cdot\nabla\,\underline{u}(\underline{x},t) + O(\eta^2)}$$

The idea is

$\underline{u}(\underline{x},t)$ is

of $\underline{X}(t;$

$\mathbb{R}^d$

$$\frac{\partial\underline{X}(t;a)}{\partial t} = \underline{u}$$

$$\underline{a} \longrightarrow \underline{X}(t$$

Hamilton's Pp'le.

$$S[\underline{X}(t;\underline{a})] = \int_{t_0}^{t_1} L\, dt \quad , \text{ where}$$

$$L = \int_{V_0} J\, \mathcal{L}(\underline{v}, \rho, p)\, d^d\underline{a}$$

where $J\mathcal{L} = \frac{1}{2} \rho_0 |\underline{v}|^2 - p(1 - J)$

$$J = \det\left(\frac{\partial \underline{X}}{\partial \underline{a}}\right),$$

The idea is that $\underline{u}(\underline{x}, t)$ is a funct

of $\underline{X}(t;\underline{a})$.

$$\frac{\partial \underline{X}(t;a)}{\partial t} = \underline{u}(\underline{X}(t;a)$$

$\underline{a} \longrightarrow \underline{X}(t;\underline{a})$ ONT

$\mathcal{L}\, dt$, where

$) \, d^d\underline{a} = \int_{V_t} \mathcal{L}(\underline{v}, \rho, p)\, d^d x$

$p\left(1 - f\right)$

$=$ const

Mass conservation:

$$\rho(\underline{x}, t)\, d^d\underline{x} = \left(\rho(\underline{x}+\underline{\eta}, t) + \delta\rho\right) d^d(\underline{x} + \underline{\eta})$$

$$= \rho(\underline{x}, t)\, d^d\underline{x} + \underline{\eta} \cdot \nabla \rho\, d^d\underline{x} + (\nabla \cdot \underline{\eta})\, \rho\, d^d\underline{x} + \delta\rho\, d^d x$$

$$\Rightarrow \boxed{\delta\rho(x, t) = -\nabla \cdot (\underline{\eta}\rho)}$$

$$\boxed{\delta\underline{u}(\underline{x}, t) = \partial_t \underline{\eta} + \underline{u} \cdot \nabla \underline{\eta} - \underline{\eta} \cdot \nabla \underline{u}}$$

$$\mathcal{L} = \frac{1}{2}\,\frac{J\rho_0}{\rho}\,|\underline{v}|^2 - p\left(J^{-1}-1\right)$$

$$\mathcal{L} = \frac{1}{2}\rho\,|\underline{v}|^2 - p\left(J^{-1}-1\right)$$

$$\rho_0\, d^d a = \rho\, d^d x = \rho J\, d^d a$$

Variation of J?

$$J = \det\left(\frac{\partial \underline{x}}{\partial \underline{a}}\right) \Rightarrow J^{new} = \det\left(\frac{\partial(\underline{x}+\underline{\eta})}{\partial \underline{a}}\right)$$

$$\frac{\partial(\underline{x}+\underline{\eta})}{\partial \underline{a}} = \frac{\partial \underline{x}}{\partial \underline{a}} + \left(\frac{\partial \underline{x}}{\partial \underline{a}}\right)\frac{\partial \underline{\eta}}{\partial \underline{x}}$$

$$\det(\mathbb{A}+\mathbb{B}) = (\det\mathbb{A})\,\det\left(\mathbb{1}+\mathbb{A}^{-1}\mathbb{B}\right)$$

$$= (\det\mathbb{A})\left(1+\mathrm{tr}\left(\mathbb{A}^{-1}\mathbb{B}\right)\right)$$

valid for "small" $\mathbb{B}$

$$\mathbb{A} = \frac{\partial x}{\partial a} \qquad \mathbb{B} = \mathbb{A}\frac{\partial \eta}{\partial x} \Rightarrow \mathrm{tr}\left(\mathbb{A}^{-1}\mathbb{B}\right) = \mathrm{tr}\left(\frac{\partial \eta}{\partial x}\right)$$

Mass conservation:

$$\rho(\underline{x},t)\,d^d\underline{x} = \left(\rho(\underline{x}+\underline{\eta},t)+\delta\rho\right)\, d^d(\underline{x}+\underline{\eta})$$

$$= \rho(\underline{x},t)\,d^d\underline{x} + \underline{\eta}\cdot\nabla\rho\, d^d\underline{x} + (\nabla\cdot\underline{\eta})\,\rho\, d^d\underline{x} + \delta\rho\, d^d x$$

$$\Rightarrow \boxed{\delta\rho(x,t) = -\nabla\cdot(\underline{\eta}\rho)}$$

$$\boxed{\delta\underline{u}(\underline{x},t) = \partial_t\underline{\eta} + \underline{u}\cdot\nabla\underline{\eta} - \underline{\eta}\,\nabla\underline{u}}$$

$$\boxed{J^{new} = J\left(1+\nabla\cdot\underline{\eta}\right)}$$

$$\mathcal{L} = \frac{1}{2} \underbrace{J^{-1}\rho_0}_{\rho} |\underline{v}|^2 - p\left(J^{-1} - 1\right)$$

$$\mathcal{L} = \frac{1}{2}\rho |\underline{v}|^2 - p\left(J^{-1} - 1\right)$$

$$\rho_0 d^d a = \rho\, d^d x = \rho J d^d a$$

$$\delta\mathcal{L} = \frac{1}{2}\delta\rho\, |\underline{v}|^2 + \rho\, \underline{v}\cdot\delta\underline{v} - \delta p\left(J^{-1}-1\right) - p\,\delta\left(J^{-1}\right)$$

$$= -\frac{1}{2}\nabla\cdot(\underline{\eta}\rho)\,|\underline{v}|^2 + \rho\,\underline{v}\cdot\left(\partial_t\underline{\eta} + \underline{u}\cdot\nabla\underline{\eta} - \underline{\eta}\cdot\nabla\underline{u}\right)$$

$$+\, p J^{-2} J\,(\nabla\cdot\underline{\eta}) - \delta p\left(J^{-1}-1\right)$$

EOM: $\frac{\delta\mathcal{L}}{\delta p}$: $\boxed{J = 1}$

Mass conservation:

$$\rho(\underline{x},t)\,d^d\underline{x} = \left(\rho(\underline{x}+\underline{\eta},t)+\delta\rho\right) d^d(\underline{x}+\underline{\eta})$$

$$= \rho(\underline{x},t)\,d^d x + \underline{\eta}\cdot\nabla\rho\, d^d\underline{x} + (\nabla\cdot\underline{\eta})\,\rho\, d^d\underline{x} + \delta\rho\, d^d x$$

$$\Rightarrow \boxed{\delta\rho(x,t) = -\nabla\cdot(\underline{\eta}\rho)}$$

$$\boxed{\delta\underline{u}(\underline{x},t) = \partial_t\underline{\eta} + \underline{u}\cdot\nabla\underline{\eta} - \underline{\eta}\,\nabla\underline{u}}$$

$$\boxed{J^{\text{New}} = J\left(1 + \nabla\cdot\underline{\eta}\right)}$$

$$\delta J = J^{\text{new}} - J = J\,(\nabla\cdot\underline{\eta})$$

$$\frac{\delta L}{\delta \eta}: + \frac{1}{2}\rho_0\, \underline{\eta}\cdot\nabla(|\underline{u}|^2) - \rho_0 \underline{\eta}\cdot\partial_t \underline{u} - \rho_0 \eta_j\, \nabla\cdot(\underline{u}\, u_j) - \rho_0\, \underline{\eta}\cdot(\nabla u_j)\, u_j - \nabla p$$

$$\frac{\delta L}{\delta \eta} = 0 \Rightarrow \boxed{\frac{1}{2}\rho_0 \nabla(|\underline{u}|^2) - \rho_0 \partial_t \underline{u} - \rho_0\, \underline{u}\cdot\nabla \underline{u} - \rho_0 \nabla\left(\frac{1}{2}|u|^2\right) - \nabla p = 0}$$

$$\delta L = \frac{1}{2}\delta\rho\, |\underline{v}|^2 + \rho\, \underline{v}\cdot\delta\underline{v} - \delta p\,(J^{-1} - 1) - p\,\delta(J^{-1})$$

$$= -\frac{1}{2}\nabla\cdot(\underline{\eta}\rho)\,|\underline{u}|^2 + \rho\,\underline{u}\cdot\left(\partial_t \underline{\eta} + \underline{u}\cdot\nabla\underline{\eta} - \underline{\eta}\cdot\nabla\underline{u}\right)$$

$$+ p J^{-2} J\,(\nabla\cdot\underline{\eta}) - \delta p\,(J^{-1} - 1)$$

$\Rightarrow$ EOM: $\frac{\delta L}{\delta p}$: $\boxed{J = 1} \Rightarrow \nabla\cdot\underline{v} = 0 \Rightarrow \rho \equiv \rho_0$ const

conservation:

$$= \left(\rho(\underline{x}+\underline{\eta}, t \ldots\right.$$

$$d^d(\underline{x} \ldots$$

$$= \rho(\underline{x}, t)\, d^d \ldots$$

$$+ \underline{\eta}\cdot\nabla\rho \ldots$$

$$+ (\nabla\cdot\underline{\eta})\,\rho\, d^d \ldots$$

$$\Rightarrow \boxed{\delta\rho(\underline{x}, t) = -\nabla\cdot(\ldots}$$

$$\boxed{\delta\underline{u}(\underline{x}, t) = \partial_t \underline{\eta} + \underline{u}\cdot\nabla\underline{\eta} - \ldots}$$

$$J^{\text{new}} = J\left(1 + \nabla\cdot\underline{\eta} \ldots\right.$$

$$\delta J = J^{\text{new}} - J = J\,(\nabla \ldots$$

$$\frac{\delta \mathcal{L}}{\delta \eta} = 0 + \frac{1}{2}\rho_0\, \underline{\eta}\cdot\nabla\left(|\underline{u}|^2\right) - \rho_0 \underline{\eta}\cdot\partial_t \underline{u} - \rho_0 \eta_j\, \nabla\cdot\left(\underline{\eta}\, u_j\right) - \rho_0\, \eta\cdot(\nabla u_j)\, u_j - \nabla p$$

$$\frac{\delta \mathcal{L}}{\delta \eta} = 0 \Rightarrow \boxed{\cancel{\frac{1}{2}\rho_0 \nabla\left(|\underline{u}|^2\right)} - \rho_0 \partial_t \underline{u} - \rho_0\, \underline{u}\, \nabla\, \underline{u} - \cancel{\rho_0 \nabla\left(\frac{1}{2}|u|^2\right)} - \nabla p = 0}$$

$$\Rightarrow \boxed{\partial_t \underline{u} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{\nabla p}{\rho_0}}$$

$$\partial_t \underline{U} + \underbrace{\mathcal{L}_{\underline{u}}\, \underline{U}} = -\nabla \pi$$

$$\underline{u}\cdot\nabla \underline{U} + U_j \nabla u^j$$

conser

$= \rho$

$+$

$+ ($

$\Rightarrow \boxed{\delta\rho(x,t) =}$

$\boxed{\delta \underline{u}(\underline{x},t) = \partial_t \underline{\eta} +}$

$\boxed{g^{new} = g\,(1}$

$\delta g = g^{\sim} - g =$

$$\frac{\delta \ell}{\delta \eta} : \; + \frac{1}{2}\rho_0\, \underline{\eta}\cdot\nabla(|\underline{u}|^2) - \rho_0 \underline{\eta}\cdot\partial_t \underline{u} - \rho_0 \eta_j\, \nabla\cdot(\underline{u}\, u_j) - \rho_0\, \underline{\eta}\cdot(\nabla u_j)\, u_j - \nabla p$$

$$\frac{\delta \ell}{\delta \eta} = 0 \Rightarrow \boxed{\cancel{\frac{1}{2}\rho_0 \nabla(|\underline{u}|^2)} - \rho_0 \partial_t \underline{u} - \rho_0\, \underline{u}\,\nabla\, \underline{u} - \rho_0 \nabla\cancel{\left(\frac{1}{2}|\underline{u}|^2\right)} - \nabla p = 0}$$

$$K = \oint_{C(t)} \underline{U}\cdot dx$$

$$\Rightarrow \boxed{\partial_t \underline{u} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{\nabla p}{\rho_0}}$$

$$\boxed{\partial_t \underline{U} + \underbrace{\pounds_{\underline{u}}\, \underline{U}}_{\underline{u}\cdot\nabla \underline{U} + U_j \nabla u^j} = -\nabla \pi}$$

Kelvin's circulation Theorem.

$\Rightarrow \delta$ ...

$\delta \underline{u}(\underline{x},$ ...

$g^{New} =$ ...

$\delta g =$ ...

$$\underline{x} \longrightarrow y(\underline{x})$$

$$u^i(\underline{x},t) \longrightarrow \tilde{u}^{j'} = \frac{\partial y^{j'}}{\partial x^k} u^k \qquad U_i = \delta_{ij} u^j$$

$$U_j(\underline{x},t) \longrightarrow \tilde{U}_{j'} = \frac{\partial x^k}{\partial y^{j'}} U_k$$

$$\boxed{\partial_t \underline{U} + \underbrace{\mathcal{L}_{\underline{u}} \underline{U}}_{\underline{u}\cdot\nabla \underline{U} + U_j \nabla u^j} = -\nabla \pi} \longrightarrow \partial_t \underline{\tilde{U}} + \mathcal{L}_{\underline{\tilde{u}}} \underline{\tilde{U}} = -\tilde{\nabla}\tilde{\pi}$$

conservation:

$$\left(\rho(\underline{x}+\underline{\eta},t)+\delta\rho\right) d^d(\underline{x}+\underline{\eta})$$

$$= \rho(\underline{x},t)\, d^d\underline{x} + \underline{\eta}\cdot\nabla\rho\, d^d\underline{x} + (\nabla\cdot\underline{\eta})\rho\, d^d\underline{x} + \delta\rho\, d^d\underline{x}$$

$$\Rightarrow \boxed{\delta\rho(\underline{x},t) = -\nabla\cdot(\underline{\eta}\rho)}$$

$$\boxed{\delta \underline{u}(\underline{x},t) = \partial_t \underline{\eta} + \underline{u}\cdot\nabla\underline{\eta} - \underline{\eta}\cdot\nabla\underline{u}}$$

$$\boxed{g^{\text{new}} = g\,(1+\nabla\cdot\underline{\eta})}$$

$$\delta g = g^{\text{new}} - g = g\,(\nabla\cdot\underline{\eta})$$