KINETIC EQUATIONS: STATISTICS AND DYNAMICS IN THE INVERSE CASCADE OF 2 D TURBULENCE

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TURBULENT CASCADES

- DIRECT CASCADE
- INVERSE CASCADE
REMINDER ON POINT VORTEX DYNAMICS
IDEAL 2D HYDRODYNAMICS

VORTEX DYNAMICS

\[ \frac{\partial}{\partial t} \omega(x, t) + u(x, t) \cdot \nabla \omega(x, t) = 0 \]

\[ u(x, t) = \int d\mathbf{x}' \omega(x', t) \mathbf{e}_z \times \frac{x - x'}{2\pi|x - x'|^2} \]

- CONSERVATION OF LAGRANGIAN VORTICITY
- BIOT-SAVART'S LAW
- POINT VORTEX
IDEAL 2D HYDRODYNAMICS

VORTEX DYNAMICS

- POINT VORTEX
- LAGRANGIAN PICTURE
- HAMILTONIAN SYSTEM

\[ \omega(x, t) = \sum_j \Gamma_j \delta(x - X_j(t)) \]

\[ u(x, t) = \sum_j \Gamma_j e_z \times \frac{x - X_j(t)}{2\pi|x - X_j(t)|^2} \]

\[ H = -\frac{1}{4\pi} \sum_{i \neq j} \Gamma_i \Gamma_j \ln|X_i - X_j| \]

\[ \dot{X}_j(t) = \sum_{k \neq j} \Gamma_k e_z \times \frac{X_j(t) - X_k(t)}{2\pi|X_j(t) - X_k(t)|^2} \]
TWO-VORTEX MOTION

- THREE POINT VORTEX MOTION
  INTEGRABLE, FOUR POINT MOTION CHAOTIC
- DISSIPATION

\[ \dot{\mathbf{X}}_1 = \Gamma_2 \mathbf{e}_z \times \frac{\mathbf{X}_1 - \mathbf{X}_2}{2\pi|\mathbf{X}_1 - \mathbf{X}_2|^2} \]

\[ \dot{\mathbf{X}}_2 = \Gamma_1 \mathbf{e}_z \times \frac{\mathbf{X}_2 - \mathbf{X}_1}{2\pi|\mathbf{X}_2 - \mathbf{X}_1|^2} \]

\[ \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2 \]

\[ \dot{\mathbf{r}} = (\Gamma_1 + \Gamma_2) \mathbf{e}_z \times \frac{\mathbf{r}}{r^2} \]

\[ \frac{d}{dt} |\mathbf{r}| = \text{const} \]
TWO-VORTEX MOTION: EFFECT OF DISSIPATION
EULERIAN VERSUS LAGRANGIAN PICTURE
with A. Daitche
THREE-VORTEX MOTION
L. ONSAGER: STATISTICS OF POINT VORTEX SYSTEMS

- POINT VORTEX SYSTEM: HAMILTONIAN SYSTEM
- STATISTICAL TREATMENT
- EQUILIBRIUM ENSEMBLE

\[ F(x_1, \ldots, x_N) = Z^{-1}(\beta)e^{-\beta H} \]

- KINETIC EQUATIONS: LIOUVILLE EQUATION, BBGKY-HIERARCHY

\[ \frac{\partial}{\partial t} + \sum_{i \neq j} \left( \Gamma_j e^z \times \frac{x_i - x_j}{2\pi|x_i - x_j|^2} \cdot \nabla_i \right) F(x_1, \ldots, x_N, t) = 0 \]
VORTICITY STATISTICS IN THE INVERSE CASCADE OF 2D TURBULENCE

with O. KAMPS, M. VOSSKUHLE
GENERATING A STATIONARY CASCADE: FLUX EQUILIBRIUM

\[
\frac{\partial}{\partial t} \omega(x, t) + u(x, t) \cdot \nabla \omega(x, t) = L(-\Delta) \omega(x, t) + F(x, t)
\]

\[
\langle F(x, t) F(x', t') \rangle = Q(|x - x'|) \delta(t - t')
\]

\[
L(-\Delta) = -\gamma - \nu(-\Delta)^\alpha
\]

- **F**: Small-Scale Stirring = Excitation of Point Vortices of Circulation (with Gaussian Statistics)
- **L**: Large-Scale Damping
MOVIES: CLUSTERING OF VORTICITY
\[ u(x, t) = \int dk e^{i k \cdot x} u_k(t) \]

\[ E(k) = 2\pi k \langle u_k u_{-k} \rangle \]
ENERGY SPECTRUM, SPECTRAL ENERGY FLUX

- CONSTANT SPECTRAL ENERGY FLUX
- O. KAMPS

\[ E(k) = C_K k^{-5/3} \]

\[ \frac{\partial}{\partial t} E(k, t) + \frac{\partial}{\partial k} \Pi(k, t) = L(k^2)E(k, t) + Q(k) \]
STRUCTURE FUNCTIONS

$\mathbf{v}_r(r,t) = \mathbf{e}_r \cdot [\mathbf{u}(x+r,t) - \mathbf{u}(x,t)]$

$S^n_r(r) = \langle v^n_r(r,t) \rangle = s_n r^{n/3}$

$S^3_r(r) = \langle v^3_r(r,t) \rangle = +s_3 r$

- SCALING BEHAVIOUR IN THE INERTIAL RANGE
- NO INTERMITTENCY
- KRAICHNAN-PREDICTIONS
VORTICITY INCREMENT STATISTICS

\[ \Omega(r, t) = \omega(x + r, t) - \omega(x, t) \]

- DESPITE NEAR-GAUSSIANITY: NO DERIVATION OF SCALING BEHAVIOUR
ANALYTICAL DERIVATION OF ENERGY SPECTRUM

• DESPITE NEAR-GAUSSIANITY: NO THEORETICAL DERIVATION OF CORRELATION FUNCTIONS

• RENORMALIZED PERTURBATION THEORIES

• RENORMALIZATION GROUP THEORIES
A.A. MIGDAL (1993): After trying for few years to do something with the Wyld approach I conclude that this is a dead end. The best bet here would be the renormalization group, which magically works in statistical physics. Those critical phenomena were close to Gaussian...There is no such luck in turbulence. The nonlinear effects are much stronger...No! These old tricks are not going to work, we have to invent the new ones

NONPERTURBATIVE TREATMENT!
WHAT IS THE NATURE OF THE TRANSPORT PROCESS IN THE INVERSE CASCADE
INVERSE CASCADE: NON-FICKIAN TRANSPORT IN SCALE

- TRANSPORT OF ENERGY UPHILL!
- STATISTICAL TREATMENT + NONLINEAR DYNAMICS: KINETIC EQUATIONS
KINETIC EQUATIONS FOR 2D-TURBULENCE: DYNAMICS AND STATISTICS

- KINETIC EQUATIONS FOR TURBULENT FIELDS
- MONIN, LUNDGREN, NOVIKOV
- SIMILAR TO BBGKY HIERARCHY OF CLASSICAL MECHANICS
- RECENT APPLICATIONS:
  - M. WILCZEK, R.F. (VORTICITY FIELD 3D DIRECT CASCADE, PHYS. REV. E (2010))
KINETIC EQUATIONS FOR TURBULENT VORTICITY

\[ f(\omega, \mathbf{x}, t) = \langle \delta(\omega - \omega(\mathbf{x}, t)) \rangle \]

\[ \frac{\partial}{\partial t} f(\omega, \mathbf{x}, t) = \langle \frac{\partial}{\partial t} \delta(\omega - \omega(\mathbf{x}, t)) \rangle \]

\[ = - \frac{\partial}{\partial \omega} \langle \dot{\omega}(\mathbf{x}, t) \delta(\omega - \omega(\mathbf{x}, t)) \rangle \]

\[ \frac{\partial}{\partial t} f(\omega(\mathbf{x}, t)) + \nabla_x \cdot \langle \mathbf{u}(\mathbf{x}, t) \delta(\omega - \omega(\mathbf{x}, t)) \rangle \]

\[ = - \frac{\partial}{\partial \omega} \nu \langle L(-\Delta)\omega(\mathbf{x}, t) \delta(\omega - \omega(\mathbf{x}, t)) \rangle - \frac{\partial}{\partial \omega} \langle F(\mathbf{x}, t) \delta(\omega - \omega(\mathbf{x}, t)) \rangle \]

- CLOSURE PROBLEM OF NONLINEAR FIELD THEORIES
- COUPLING TO TWO-POINT FUNCTIONS (HIERARCHY)
- INTRODUCTION OF CONDITIONAL EXPECTATIONS
KINETIC EQUATIONS FOR TURBULENT VORTICITY

\[
\frac{\partial}{\partial t} f(\omega(x, t)) + \nabla_x \cdot \langle u(x, t) \delta(\omega - \omega(x, t)) \rangle = -\frac{\partial}{\partial \omega} \nu \langle L(-\Delta)\omega(x, t) \delta(\omega - \omega(x, t)) \rangle - \frac{\partial}{\partial \omega} \langle F(x, t) \delta(\omega - \omega(x, t)) \rangle
\]

- CONDITIONAL EXPECTATIONS

\[
\langle u(x, t) \delta(\omega - \omega(x, t)) \rangle = \langle u(x, t) | \omega, x \rangle f(\omega, x, t)
\]

\[
\langle (Diss + Forc) \delta(\omega - \omega(x, t)) \rangle = \langle \mu(x | \omega, x) \rangle f(\omega, x, t)
\]
TWO-POINT VORTICITY STATISTICS $f(\omega_1, x_1; \omega_2, x_2; t)$

- Kinetic Equation

$$\left[ \frac{\partial}{\partial t} + \nabla_{x_1} \cdot \langle \mathbf{u}(x_1) | \omega_1, x_1; \omega_2, x_2 \rangle + \nabla_{x_2} \cdot \langle \mathbf{u}(x_1) | \omega_1, x_1; \omega_2, x_2 \rangle \right] f(\omega_1, x_1; \omega_2, x_2; t)$$

$$= -\left[ \frac{\partial}{\partial \omega_1} \langle \mu(x_1 | \omega_1, x_1; \omega_2, x_2) \rangle + \frac{\partial}{\partial \omega_2} \langle \mu(x_2 | \omega_1, x_1; \omega_2, x_2) \rangle \right] f(\omega_1, x_1; \omega_2, x_2; t)$$

- First Order Partial Differential Equation
- Solvable by Methods of Characteristics
- CONDITIONAL EXPECTATIONS ACCESSIBLE BY DIRECT NUMERICAL SIMULATIONS
TWO-POINT VORTICITY STATISTICS

\[ f(\omega_1, x_1; \omega_2, x_2; t) \]

- **Kinetic Equation**

\[
\frac{\partial}{\partial t} + \nabla_{x_1} \cdot \langle u(x_1) | \omega_1, x_1; \omega_2, x_2 \rangle + \nabla_{x_2} \cdot \langle u(x_1) | \omega_1, x_1; \omega_2, x_2 \rangle] f(\omega_1, x_1; \omega_2, x_2; t)
\]

\[
= - \left[ \frac{\partial}{\partial \omega_1} \langle \mu(x_1) | \omega_1, x_1; \omega_2, x_2 \rangle + \frac{\partial}{\partial \omega_2} \langle \mu(x_2) | \omega_1, x_1; \omega_2, x_2 \rangle \right] f(\omega_1, x_1; \omega_2, x_2; t)
\]

- **Characteristic Equations**

\[
\dot{x}_1 = \langle u(x_1) | \omega_1, x_1; \omega_2, x_2 \rangle
\]

\[
\dot{x}_1 = \langle u(x_2) | \omega_1, x_1; \omega_2, x_2 \rangle
\]

\[
\dot{\omega}_1 = \langle \mu(x_1) | \omega_1, x_1; \omega_2, x_2 \rangle
\]

\[
\dot{\omega}_2 = \langle \mu(x_2) | \omega_1, x_1; \omega_2, x_2 \rangle
\]
EXTENSION TO N-POINT STATISTICS

\[ \dot{x}_i = \langle U(x_i) \rvert \{\omega_k, x_k\} \rangle \quad \dot{\omega}_i = \langle \mu(x_i) \rvert \{\omega_k, x_k\} \rangle \]

- **N TO INFINITY: APPROACHING LAGRANGIAN DESCRIPTION OF FLUID MOTION**

\[ \dot{X}(y, t) = \int dy' \Omega(y', t)e_z \times \frac{X(y, t) - X(y', t)}{2\pi|X(y, t) - X(y', t)|^2} \]

\[ \dot{\Omega}(y, t) = [\nu \Delta \omega(x, t) + F(x, t)]_{x = X(y, t)} \]
CAN WE DETERMINE CONDITIONAL EXPECTATIONS BY AB INITIO CALCULATIONS
A FIRST GUESS ON CONDITIONAL VELOCITY FIELDS:
ON THE WAY TO SUBGRID-MODELING

● CONDITIONAL STATISTICS: ASSUME GAUSSIAN STATISTICS, CORRELATION FUNCTION $C(x-x')$

$$\langle U(x, t)|\omega_1, x_1; \omega_2, x_2 \rangle = \int dx' \langle \omega(x')|\omega_1, x_1; \omega_2, x_2 \rangle e_z \times \frac{x - x'}{2\pi|x - x'|^2}$$

$$\langle \omega(x)|\omega_1, x_1; \omega_2, x_2 \rangle = \int d\omega' \omega' p(\omega', x'|\omega_1, x_1; \omega_2, x_2)$$

● CONDITIONAL VORTICITY: SUPERPOSITION OF CIRCULAR VORTICES

$$\langle \omega(x)|\omega_1, x_1; \omega_2, x_2 \rangle = \left[ C(x - x_1)C^{-1}(0)\omega_1 + C(x - x_2)C^{-1}(0)\omega_2 \right]$$
DRESSED VORTEICES

\[ \langle U(x)|\omega_1, x_1; \omega_2, x_2 \rangle = \omega_1 e_z \times \nabla \chi(x - x_1) + \omega_2 e_z \times \nabla \chi(x - x_2) \]

\[ \Delta \chi(x - x_1) = C^{-1}(0)C(x - x_1) \]

- **BARE POINT VORTEX:**
  \[ C^{-1}(0)C(r) \rightarrow \delta(r) \quad U = e_z \times \frac{r}{r^2} \]

- **DRESSED VORTEX:**
  \[ U = e_z \times \frac{r}{h(r)} \]

- **DIFFERENT Biot-Savart's Law**

- **Landau Quasi-Particles**
CONDITIONAL GAUSSIAN APPROXIMATION

- MEASUREMENT

\[ \langle \omega(x,y)|\omega_1 \approx 3.7\sqrt{\langle \omega^2 \rangle}, \omega_2 = -\omega_1, r \approx 4\lambda \rangle/\sqrt{\langle \omega^2 \rangle} \]

- APPROXIMATION

\[ \langle \omega(x,y)|\omega_1 \approx 3.7\sqrt{\langle \omega^2 \rangle}, \omega_2 = -\omega_1, r \approx 4\lambda \rangle/\sqrt{\langle \omega^2 \rangle} \]
CONDITIONAL GAUSSIAN APPROXIMATION

- MEASUREMENT

\[ \langle \omega(x,y) \mid \omega_1 \approx 1.9 \sqrt{\langle \omega^2 \rangle}, \omega_2 = -\omega_1, r \approx \lambda \rangle / \langle \omega^2 \rangle \]

- APPROXIMATION

\[ \langle \omega(x,y) \mid \omega_1 \approx 1.9 \sqrt{\langle \omega^2 \rangle}, \omega_2 = -\omega_1, r \approx \lambda \rangle / \langle \omega^2 \rangle \]
FAILURE OF THE GAUSSIAN APPROXIMATION

\[
\begin{align*}
\dot{x}_1 &= e_z \times \frac{x_1 - x_2}{h(|x_1 - x_2|)} \omega_2 \\
\dot{x}_2 &= e_z \times \frac{x_2 - x_1}{h(|x_2 - x_1|)} \omega_1
\end{align*}
\]

\[
\begin{align*}
\omega_1 &= -\gamma \omega_1 + \eta_1(t) \\
\omega_2 &= -\gamma \omega_1 + \eta_2(t)
\end{align*}
\]

\[
\frac{\partial}{\partial t} |x_1 - x_2| = 0
\]

- NO CASCADE, NO ENERGY FLUX
- DEVIATIONS FROM GAUSSIANITY!
- INTERACTION OF QUASI-PARTICLES

\[
S_r^3(r) = 0
\]
EXTENSION OF THE GAUSSIAN APPROXIMATION

\[ \dot{X}_1 = e_z \times \frac{X_1 - X_2}{h(|X_1 - X_2|)} \omega_2 + e_r U(|X_1 - X_2|, \omega_1, \omega_2) \]

\[ \dot{X}_2 = e_z \times \frac{X_2 - X_1}{h(|X_2 - X_1|)} \omega_1 + e_r U(|X_2 - X_1|, \omega_2, \omega_1) \]

- ATTRACTIVE, REPULSIVE INTERACTION
- BREAKING OF PARITY-TIME-INVARIANCE

\[ U(r, \omega_1, \omega_2) = U(r, -\omega_1, -\omega_2) \]
LANDAU'S QUASI-PARTICLES
QUASI-VORTICIES

- TWO-POINT PROBABILITY DISTRIBUTION
- KINETIC EQUATION (REDUCED EULERIAN STATISTICS)
- CHARACTERISTIC EQUATION (AVERAGED LAGRANGIAN STATISTICS)
- QUASI-VORTICIES:
  - SCREENED BIOT-SAVARTŚ LAW
  - RELATIVE MOTION (LINKED TO CASCADE, KARMAN-HOWARTH RELATION)
- WHERE DOES THE GLUING FORCE COME FROM
STATISTICS OF RELATIVE MOTION FROM DNS
RELATIVE MOTION: DYNAMICS OF VORTICITY INCREMENTS

- VORTICITY INCREMENT

\[ \Omega = \omega(x + r, t) - \omega(x, t) \]

\[ \frac{\partial}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} r \langle U(r)|\Omega \rangle \rangle f(\Omega, r, t) = \frac{\partial}{\partial \Omega} \mu(r, \Omega) f(\Omega, r, t) \]

- CONDITIONAL LONGITUDINAL VELOCITY INCREMENT

\[ \langle U(r)|\Omega \rangle \]

- FLUX IN THE CASCADE:
WITH A LITTLE HELP FROM
DIRECT NUMERICAL SIMULATION

\[ U(r, \Omega) = g(r) \left[ \Omega^2 - \langle \Omega(r)^2 \rangle \right] \]

\[ \Omega = \omega_1 - \omega_2 \]
\[ \langle U(r) | \Omega \rangle = g(r) \left[ \Omega^2 - \langle \Omega(r) \rangle^2 \right] \]

\[ \mu(r, \Omega) = \mu(r) \Omega \]
COMPUTERS ARE BORING: THEY ONLY GIVE YOU ANSWERS
DYNAMICS OF VORTICITY INCREMENTS

- KINETIC EQUATION
  \[
  \frac{\partial}{\partial t} f(\Omega, r, t) + \frac{\partial}{\partial r} g(r) \left[ \Omega^2 - \langle \Omega (r, t)^2 \rangle \right] f(\Omega, r, t) = -\frac{\partial}{\partial \Omega} \mu(r) \Omega f(\Omega, r, t)
  \]

- CHARACTERISTIC EQUATION
  \[
  \dot{r} = g(r) \left[ \Omega^2 - \langle \Omega (r, t)^2 \rangle \right]
  \]
  \[
  \dot{\Omega} = \mu(r) \Omega
  \]
SOLUTION FOR THE VORTICITY PDF

\[ g(r)(\Omega^2 - \langle \Omega^2(r) \rangle) \frac{\partial}{\partial r} f(\Omega, r) + g'(r)(\Omega^2 - \langle \Omega^2(r) \rangle) f(\Omega, r) \]

\[ = \frac{\partial}{\partial \Omega} \mu(r) \Omega f(\Omega, r) \]

• Gaussian Solution at \( g'(r) = 0 \)

\[ f(\Omega, r) = Z^{-1} e^{-\frac{g'(r)}{2\mu(r)} \Omega^2} \]

\[ \langle \Omega^2(r) \rangle = \frac{\mu(r)}{g'(r)} \]
STOCHASTIC INTERPRETATION
OF CHARACTERISTICS: RATCHET EFFECT

\[ \dot{r} = g(r)\left[\Omega^2 - \langle \Omega^2(r) \rangle \right] + \eta \]
\[ = - \frac{\partial}{\partial r} V(r, \Omega) + \eta \]
\[ \dot{\Omega} = -\gamma \Omega + F \]

- DIRECTED TRANSPORT FROM UNCORRELATED NOISE!
- RATCHET!
INVERSE CASCADE WITHIN A SIMPLE VORTEX MODEL

(TOGETHER WITH J. FRIEDRICH)

WHERE DOES THE RELATIVE VELOCITY FIELD U COMES FROM?
ORIGIN OF INVERSE CASCADE: BEYOND LINEAR GAUSSIAN APPROXIMATION

- EXTENSION TO DRESSED ELLIPTICAL VORTICES
VORTEX THINNING:
INNER DEGREE'S OF FREEDOM
GENERATING ELLIPTICAL VORTICES

ELLIPtical VORTEX: TWO INELASTICALLY COUPLED POINT VORTICES (EFFECT OF SHEAR)
POINT VORTEX PAIR MODEL FOR INVERSE CASCADE

\[ \dot{x}_i = \sum_j \Gamma_j e_z \times \frac{x_i - x_j}{|x_i - x_j|^2} + \gamma(D_0 - D_{i,i+1}) \frac{x_i - x_{i+1}}{|x_i - x_{i+1}|} \]

\[ D_{ik} = |x_i - x_k| \]

- NONPERTURBATIVE PART: ENERGY INPUT AT SMALL SCALES D_0
- NONLINEAR DYNAMICS TREATMENT
- ACTIVE FLUIDS, ACTIVE PARTICLES
INVERSE CASCADE IN POINT VORTEX SYSTEMS
ONSAGER – HAMILTONIAN POINT VORTEX SYSTEM
INVERSE CASCADE IN POINT VORTEX SYSTEMS
CLUSTER FORMATION OF ELLIPTICAL VORTICES:

\[
\frac{d}{dt} R(t) = \frac{c d^4}{\gamma} \frac{(\Gamma_1 + \Gamma_2)^2}{R^5(t)}
\]

• ATTRACTION OF ROTORS BY A LARGE VORTEX AT THE ORIGIN
• CLUSTER PHYSICS!
• RESULTS OBTAINED BY AVERAGING TECHNIQUES (fast rotations, interesting nonlinear dynamics, J. Friedrich, R. F. ArXiv (2012))
TWO PAIRS OF ELLIPTICAL VORTICES: GLUING FIELD OF THE CASCADE

\[
\frac{d}{dt} R(t) = \frac{c d^4}{y} \frac{(\Gamma_1 + \Gamma_2)^2}{R^5(t)}
\]

- ATTRACTION BETWEEN VORTICES WITH THE SAME CIRCULATION
- NO INTERACTION BETWEEN VORTICES WITH OPPOSITE CIRCULATION
- RESULT OBTAINED BY AVERAGING TECHNIQUES (fast rotations, interesting nonlinear dynamics)
RECENT WORK ON INVERSE CASCADE BY POINT VORTEX DYNAMICS

\[ \omega = \sum_j \Gamma_j N e^{-(x-x_j(t))C_j^{-1}(t)(x-x_j(t))} \]

\[ \dot{x}_i = \sum_j \Gamma_j u(x_i - x_j) + \sum_j \Gamma_j \nabla_i (C_i + C_j) \nabla_i u(x_i - x_j) \]

\[ \dot{C}_i = C_i \nabla U(x_i) + \nabla U(x_i) C_i + \gamma (C_0 - C_i(t)) \]

- Approximate derivation of evolution equation for Position and Shape from Navier-Stokes
- Relationship to Lagrangian Coherent Structures
- Instanton Calculations (Solution of instanton equations by elliptical point vortex ansatz, Variational ansatz with elliptical vortices for MSR action)
SUMMARY

TWO-POINT STATISTICS OF THE INVERSE CASCADE

- LANDAU QUASI-VORTICES AND THEIR INTERACTION

- ELLIPTICAL DEFORMATIONS OF THE VORTICES: ATTRACTION OF DRESSED VORTICES

- PAIR POINT VORTEX MODEL FOR INVERSE CASCADE

- SUBGRID MODEL FOR 2D TURBULENCE