Lagrange versus Euler for turbulent flows Workshop,

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Lagrangian dynamics of the velocity gradient tensor in isotropic turbulence

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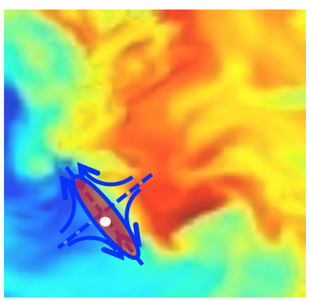
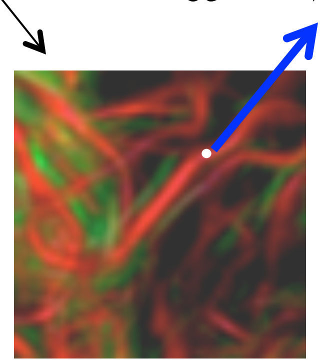
The velocity gradient tensor
Phenomenology (incompressible, NS):

$$A_{ij} = \frac{\partial u_i}{\partial x_j} = S_{ij} + \Omega_{ij}$$

$$\omega = \nabla \times \mathbf{u}$$



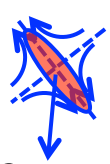
Strain-rate tensor: eigen-values, eigen-vectors



Rotation tensor Vorticity vector

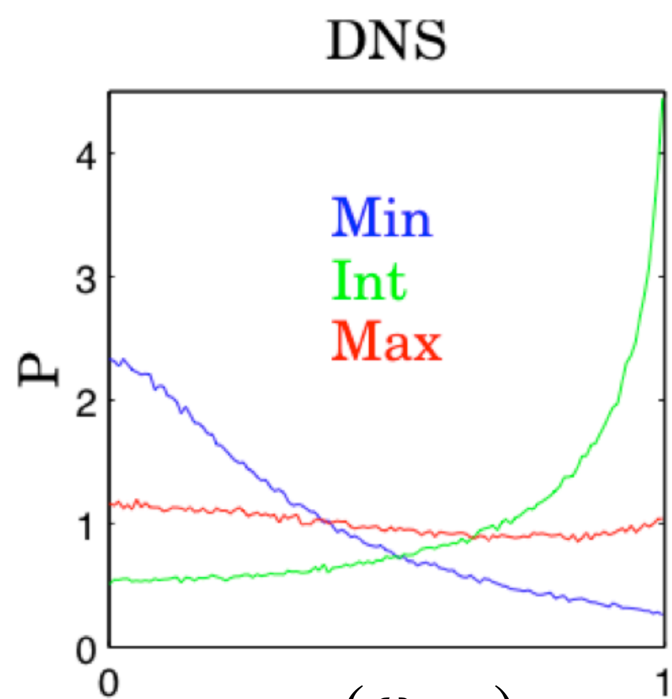
$\mathbf{e}_1 \quad (\lambda_1 > 0)$

$\mathbf{e}_3 \quad (\lambda_3 < 0)$



$\mathbf{e}_2$

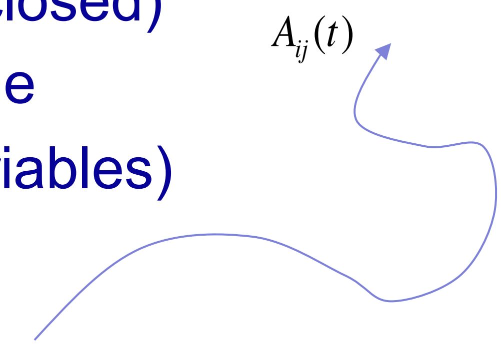
Preferential alignment of vorticity with intermediate strain-rate eigenvector (Ashurst et al. 1987):



The velocity gradient tensor

$$A_{ij} = \frac{\partial u_i}{\partial x_j} = S_{ij} + \Omega_{ij} \qquad \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial t} + \frac{\partial u_k u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + g_i \right)$$

System of 9 (8) ODEs (not closed)
if viewed in Lagrangian frame
(dependent on non-local variables)



$$\frac{dA_{11}}{dt} = \ldots$$

$$\frac{dA_{12}}{dt} = \ldots$$

$$\frac{dA_{13}}{dt} = \ldots$$

...

...

$$\frac{dA_{33}}{dt} = \ldots$$

Theme:

Could such a low-order model
predict statistics of A_{11}, A_{12},.. ?

- Geometry (alignments)
- Skewness $\langle A_{11}^3 \rangle / \langle A_{11}^2 \rangle^{3/2}$
- Anomalous scaling as function of Re

  $\langle A_{11}^p \rangle \sim Re^{\zeta(p)}$ (Nelkin, 1990; etc..)

The velocity gradient tensor

Outline:

- Review some existing models (RE, etc.)
- Describe in <u>detail</u> one model we have worked on in particular (with Chevillard)
- Some successes (eg reproducing some recent Göttingen-Lyon results on time correlations).
- Some “challenges” (=failures!)
- One option: matrix shell model (with Luca etc)
- Some further observations (with Huidan Yu)
- No conclusions other than “more work needed”, “hopefully you are interested in this”, etc., etc.

$$\frac{dA_{11}}{dt} = \ldots$$

$$\frac{dA_{12}}{dt} = \ldots$$

$$\frac{dA_{13}}{dt} = \ldots$$

$$\ldots$$

$$\ldots$$

$$\frac{dA_{33}}{dt} = \ldots$$

Collaborators:

Today's research results mainly by:

Laurent Chevillard (now Lyon),
also with L Biferale & F. Toschi
Huidan Yu (now Purdue Indianapolis)
Marco Martins-Afonso (Toulouse)

The Turbulence Database Group:

Kalin Kanov (CS PhD student)
Jason Graham (ME PhD student)
Chichi Lalescu (Appl. Math. postdoc)

Randal Burns (CS)
Greg Eyink (Applied Math)
Alex Szalay (Physics & Astronomy)
Ethan Vishniac (McMaster)
Shiyi Chen (Beijing U)

Eric Perlman (former CS PhD student)
Hussein Aluie (frormer Applied Math.)
Laurent Chevillard (former Postdoc ME)
Marco Martins-Afonso (former Postdoc)
Yi Li (former ME PhD student)
Minping Wan (former ME PhD student)
Huidan Yu (former Postdoc ME)
Tamás Budavári (Research Assoc)

Suzanne Werner,
Victor Paul,
Jan v.Berg

The velocity gradient tensor

$$A_{ij} = \frac{\partial u_i}{\partial x_j} = S_{ij} + \Omega_{ij}$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial t} + \frac{\partial u_k u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + g_i\right)$$

$$\frac{dA_{ij}}{dt} = -\underbrace{\left(A_{iq}A_{qj} - \frac{1}{3}A_{mn}A_{nm}\delta_{ij}\right)}_{\text{Self-stretching}} \underbrace{-\left(\frac{\partial^2 p}{\partial x_i \partial x_j} - \frac{1}{3}\nabla^2 p \delta_{ij}\right) + \nu \frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}}_{\text{unclosed}} + \underbrace{W_{ij}}_{\text{forcing}}$$

The velocity gradient tensor

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Restricted Euler Equation

Viellefosse 1982, Cantwell 1992:

$$\frac{dA_{ij}}{dt} = -\left(A_{iq}A_{qj} - \frac{1}{3}A_{mn}A_{nm}\delta_{ij} \right) \qquad A_{ii} = \frac{\partial u_i}{\partial x_i} = 0$$

System of 8 independent ODEs
if viewed in Lagrangian frame

The velocity gradient tensor

Restricted Euler Equation
Viellefosse 1982, Cantwell 1992:

$$\frac{dA_{ij}}{dt} = -\left(A_{iq}A_{qj} - \frac{1}{3}A_{mn}A_{nm}\delta_{ij} \right)$$

FULL ANALYTICAL SOLUTION (Cantwell 1992): $a_{ij}(\tau) = c_{ij} f_1(r(\tau)) + d_{ij} f_2(r(\tau))$

$$a_{ij} = \frac{A_{ij}}{|Q_0|}, \qquad r = \frac{R}{|Q_0|^{3/2}}, \qquad q_0 = sign(Q_0)$$

$$c_{ij} = -a_{ij}(0) q_0^2 f_2'(r_0) + \left(\frac{3}{2} a_{ik}(0) a_{kj}(0) + q(0)\delta_{ij} \right) f_2(r_0)$$

$$d_{ij} = -a_{ij}({}_0) q_0^2 f_1'(r_0) - \left(\frac{3}{2} a_{ik}(0) a_{kj}(0) + q(0)\delta_{ij} \right) f_1(r_0)$$

$$Q_0 > 0: \quad f_1^+(r) = \frac{1}{2}\left[h(r)^{\frac{1}{3}} + k(r)^{\frac{1}{3}} \right], \quad f_2^+(r) = \frac{1}{\sqrt{3}}\left[h(r)^{\frac{1}{3}} - k(r)^{\frac{1}{3}} \right], \quad h(r) = 1 + \frac{3\sqrt{3}}{2} r, \quad k(r) = 1 - \frac{3\sqrt{3}}{2} r$$

$$Q_0 < 0: \quad f_1^-(r) = \left(1 + \frac{27}{4} r^2 \right)^{\frac{1}{6}} \cos\theta, \quad f_1^-(r) = \frac{2}{\sqrt{3}}\left(1 + \frac{27}{4} r^2 \right)^{\frac{1}{6}} \sin\theta, \qquad \theta = \frac{1}{3}\tan^{-1}\left(\frac{3\sqrt{3}r}{2} \right)$$

The velocity gradient tensor

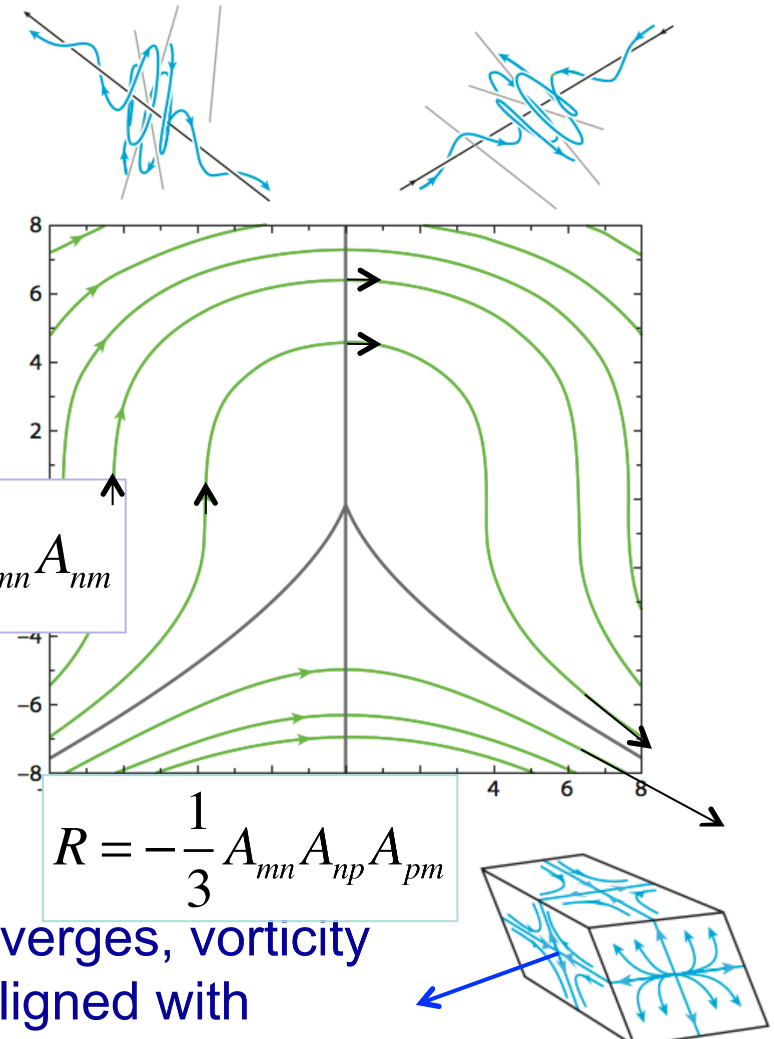
Restricted Euler Equation

Viellefosse 1982, Cantwell 1992:

$$\frac{dA_{ij}}{dt} = -\left(A_{iq}A_{qj} - \frac{1}{3}A_{mn}A_{nm}\delta_{ij} \right)$$

$$a_{ij}(\tau) = c_{ij}f_1(r(\tau)) + d_{ij}f_2(r(\tau))$$

$$Q = -\frac{1}{2}A_{mn}A_{nm}$$



$$R = -\frac{1}{3}A_{mn}A_{np}A_{pm}$$

But quick, finite-time **divergence** to infinity (scalar analogue):

$$\frac{dy}{dt} = -y^2 \Rightarrow y = \frac{1}{y_0^{-1} - t}$$

Before system diverges, vorticity becomes aligned with intermediate eigenvector

The velocity gradient tensor

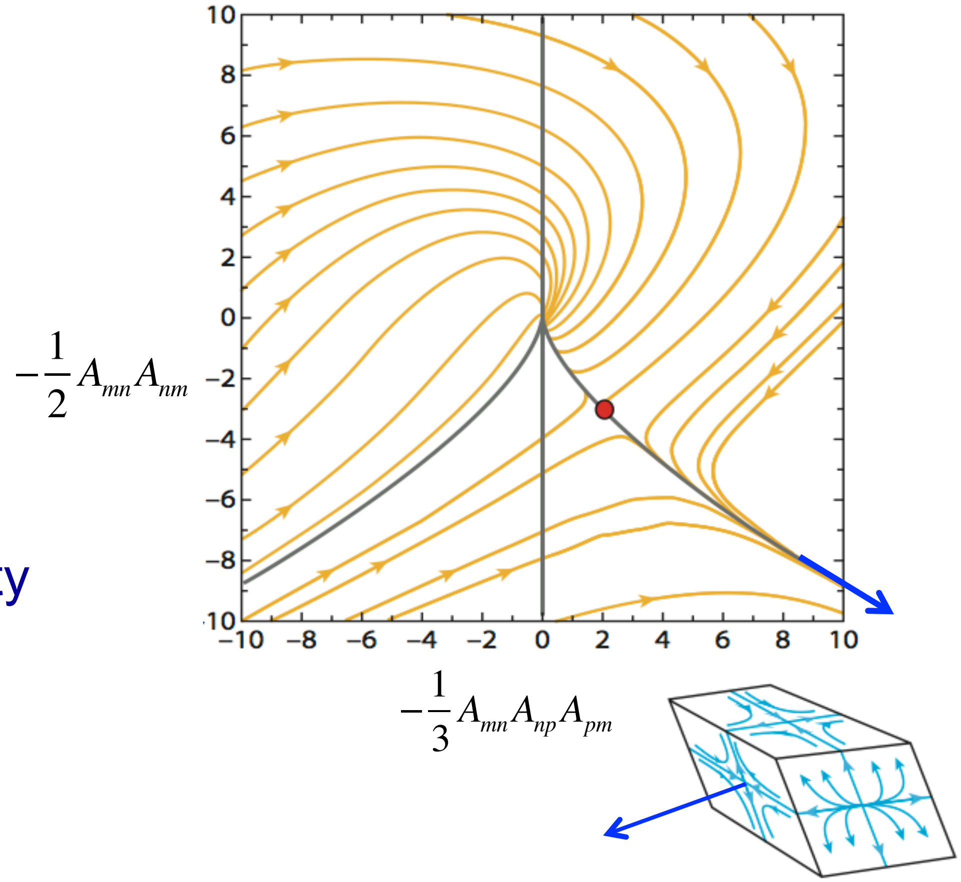
Restricted Euler Equation with linear damping

Martin et al. 1998:

$$\frac{dA_{ij}}{dt} = -\left(A_{iq}A_{qj} - \frac{1}{3}A_{mn}A_{nm}\delta_{ij} \right) - \frac{1}{T}A_{ij}$$

$$A_{ii} = \frac{\partial u_i}{\partial x_i} = 0$$

Still, many initial conditions
have finite-time **divergence** to infinity



The velocity gradient tensor

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Still, many initial conditions
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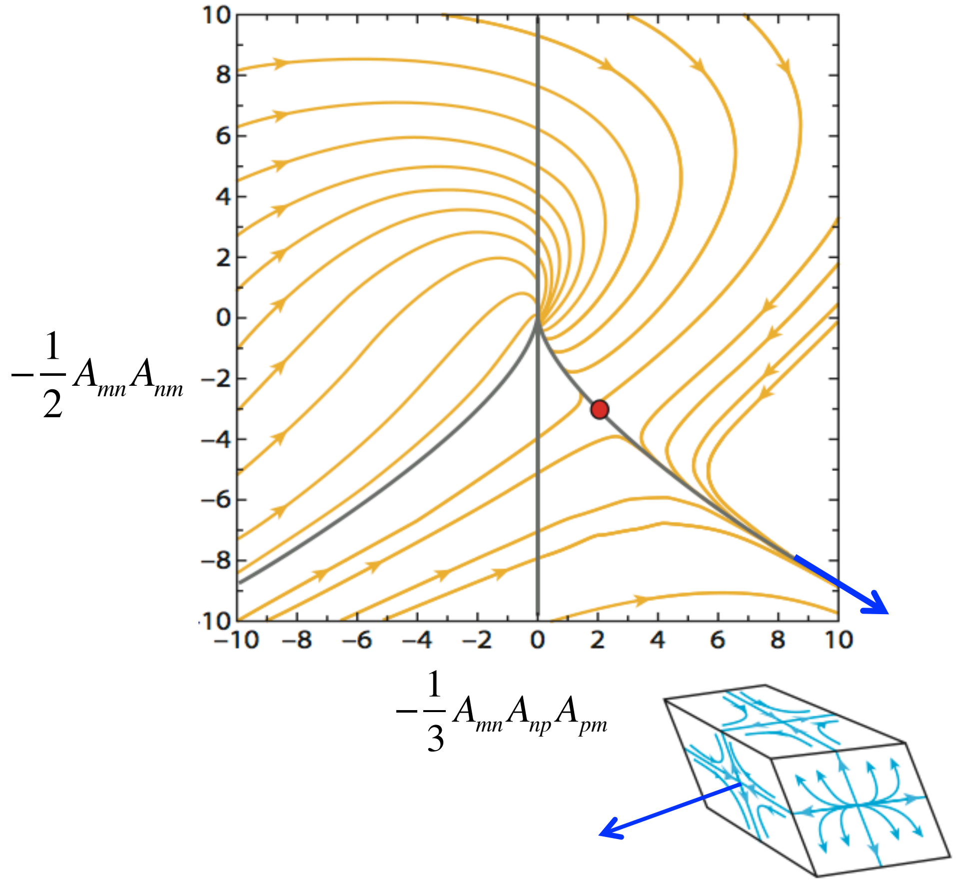
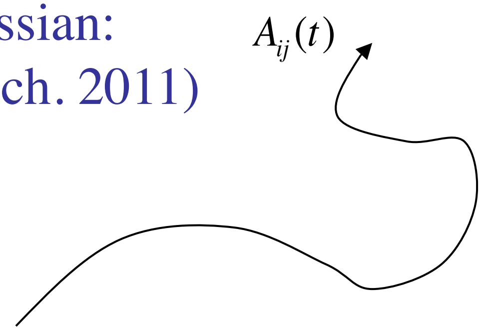
Goal: develop **Lagrangian** model for missing physics keeping simplicity of 8 ODEs (or SDE if with stochastic ingredients)



The velocity gradient tensor
Models:

$$\frac{dA_{ij}}{dt} = -\underbrace{\left(A_{iq}A_{qj} - \frac{1}{3}A_{mn}A_{nm}\delta_{ij} \right)}_{\text{Self-stretching}} \underbrace{- \left(\frac{\partial^2 p}{\partial x_i \partial x_j} - \frac{1}{3}\nabla^2 p \delta_{ij} \right) + \nu \frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}}_{\text{unclosed}} + \underbrace{W_{ij}}_{\text{forcing}}$$

Modeling the pressure Hessian:
(see CM Annu. Rev. Fluid Mech. 2011)



The velocity gradient tensor
Models:

$$\frac{dA_{ij}}{dt} = -\underbrace{\left(A_{iq}A_{qj} - \frac{1}{3}A_{mn}A_{nm}\delta_{ij}\right)}_{\text{Self-stretching}} \underbrace{-\left(\frac{\partial^2 p}{\partial x_i \partial x_j} - \frac{1}{3}\nabla^2 p\delta_{ij}\right) + \nu\frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}}_{\text{unclosed}} + \underbrace{W_{ij}}_{\text{forcing}}$$

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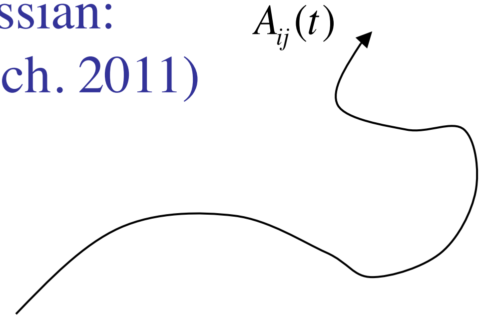


1. Stochastic diffusion model with prescribed log-normal dissipation (Girimaji & Pope 1990)

$$dA_{ij} = \left(-A_{ik}A_{kj} - \frac{1}{3}A_{mn}A_{nm}\delta_{ij} - A_{ij}\left(\frac{1}{2}\ln(A_{mn}A_{mn}) - \frac{A_{lk}N_{lk}}{A_{mn}A_{mn}} - \frac{7}{2}\frac{\sigma_\epsilon^2}{2\tau_K}\right)\right)dt + D_{ijkl}dW_{kl}$$

Gaussianity and the variance of log(dissipation) must be prescribed

The velocity gradient tensor
Models:

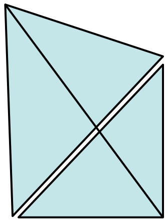
Modeling the pressure Hessian:

2. The Tetrad Model (Chertkov, Pumir, Shraiman 1999)

$$\frac{dA_{ij}}{dt} = -(1-\alpha)\left(A_{ik}A_{kj} - \Pi_{ij}A_{mn}A_{nm}\right) + \eta_{ij},$$

$$\frac{dg_{ij}}{dt} = g_{ik}A_{kj} + g_{jk}A_{ki} + \beta\sqrt{A_{mn}A_{mn}}\left(g_{ij} - \frac{1}{3}g_{kk}\delta_{ij}\right),$$

$$\Pi_{ij} = \frac{(\mathbf{g}^{-1})_{ij}}{(\mathbf{g}^{-1})_{kk}}$$

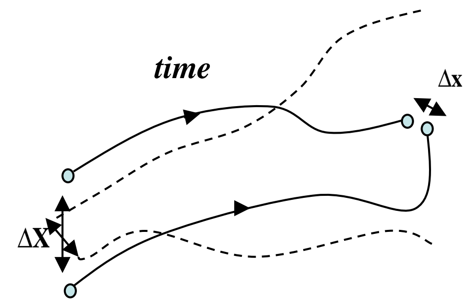


8+6 SDE – continuous evolution
(shapes keep evolving, no stationary statistics)

The velocity gradient tensor
Models:

3. Lagrangian Linear Diffusion Model (Jeong & Girimaji 2003)

Euler-Lagrange change of variables for gradients:



$$\frac{\partial F}{\partial x_k} = \frac{\partial F}{\partial X_m}\frac{\partial X_m}{\partial x_k} = \frac{\partial F}{\partial X_m}(\mathbf{D}^{-1})_{mk} \qquad D_{ij} = \frac{\partial x_j}{\partial X_i}$$

$$\nu \frac{\partial^2 A_{ij}}{\partial x_k \partial x_k} \approx \frac{\nu}{3}(\mathbf{C}^{-1})_{mn}\frac{\partial^2 A_{ij}}{\partial X_m \partial X_n}, \qquad \mathbf{C} = \mathbf{D}\mathbf{D}^T$$

Cauchy – Green tensor

$$\frac{d\mathbf{D}}{dt} = \mathbf{D}\mathbf{A}$$

$$(\mathbf{C}^{-1})_{mn} \approx \frac{1}{3}\delta_{mn}(\mathbf{C}^{-1})_{kk} \quad \text{(neglect "geometry")}$$

$$\nu\partial^2 A_{ij} / \partial X_m^2 \approx -A_{ij} / \tau_L$$

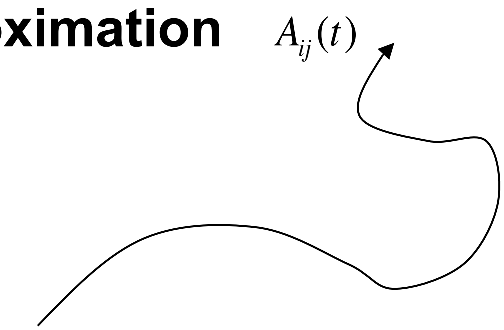
$$\frac{dA_{ij}}{dt} = -\left(A_{ik}A_{kj} - \frac{1}{3}\delta_{ij}A_{mn}A_{nm}\right) - \frac{(\mathbf{C}^{-1})_{kk}}{3\tau_L}\left(1 - \frac{\varepsilon_s}{A_{mn}A_{mn}}\right)A_{ij}.$$

8 + 6 SDE –
continuous evolution
(**C** keep evolving,
no stationary statistics)

The velocity gradient tensor
Model:

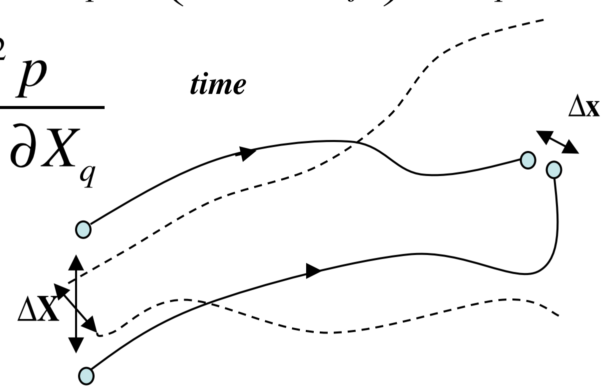
$$\frac{dA_{ij}}{dt} = -\underbrace{\left(A_{iq}A_{qj} - \frac{1}{3}A_{mn}A_{nm}\delta_{ij} \right)}_{\text{Self-stretching}} \underbrace{- \left(\frac{\partial^2 p}{\partial x_i \partial x_j} - \frac{1}{3}\nabla^2 p \delta_{ij} \right) + \nu \frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}}_{\text{unclosed}} + \underbrace{W_{ij}}_{\text{forcing}}$$

4. Lagrange-Euler Recent Fluid Deformation Approximation
Chevillard & CM 2006,2007):

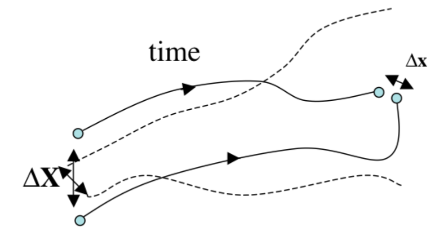


$$\frac{\partial^2 p}{\partial x_i \partial x_j} = \frac{\partial X_p}{\partial x_i}\frac{\partial X_q}{\partial x_j} \quad \frac{\partial^2 p}{\partial X_p \partial X_q} + \left(\frac{\partial}{\partial x_i}\frac{\partial X_q}{\partial x_j} \right)\frac{\partial p}{\partial X_q}$$

$$\frac{\partial^2 p}{\partial x_i \partial x_j} \approx \frac{\partial X_p}{\partial x_i}\frac{\partial X_q}{\partial x_j} \quad \frac{\partial^2 p}{\partial X_p \partial X_q}$$



A. Assume that Lagrangian pressure Hessian is isotropic if time-delay τ is long enough for memory loss of dispersion process



$$\frac{\partial^2 p}{\partial x_i \partial x_j} \approx \frac{\partial X_p}{\partial x_i}\frac{\partial X_q}{\partial x_j}\frac{\partial^2 p}{\partial X_p \partial X_q}$$

$$\frac{\partial^2 p}{\partial X_p \partial X_q} \approx \frac{1}{3}\frac{\partial^2 p}{\partial X_k \partial X_k}\delta_{pq}$$

$$\frac{\partial^2 p}{\partial x_i \partial x_j} \approx (\mathbf{C}^{-1})_{ij}\frac{1}{3}\frac{\partial^2 p}{\partial X_k \partial X_k}$$

???

Deformation tensor: $$D_{ij} = \frac{\partial x_j}{\partial X_i}$$

Cauchy-Green tensor: $$C_{ij} = D_{ik}D_{jk}$$

Inverse: $$(\mathbf{C}^{-1})_{ij} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}$$

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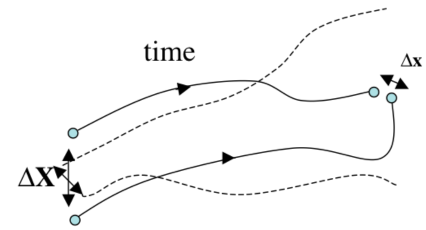

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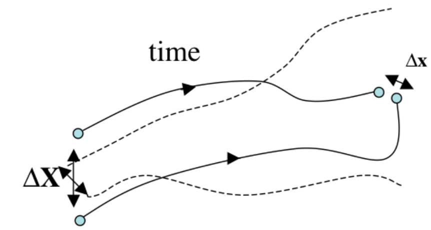
Poisson constraint: $\frac{\partial^2 p}{\partial x_i \partial x_i} = -A_{iq}A_{qi}$

$$(\mathbf{C}^{-1})_{ii}\ \frac{1}{3}\frac{\partial^2 p}{\partial X_k \partial X_k} = -A_{iq}A_{qi}$$

$$\frac{1}{3}\frac{\partial^2 p}{\partial X_k \partial X_k} = -\frac{A_{iq}A_{qi}}{(\mathbf{C}^{-1})_{ii}}$$

A. Assume that Lagrangian pressure Hessian is isotropic if time-delay τ is long enough for memory loss of dispersion process



$$\frac{\partial^2 p}{\partial x_i \partial x_j} \approx \frac{\partial X_p}{\partial x_i}\frac{\partial X_q}{\partial x_j} \quad \frac{\partial^2 p}{\partial X_p \partial X_q}$$

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$$(\mathbf{C}^{-1})_{ii} \quad \frac{1}{3}\frac{\partial^2 p}{\partial X_k \partial X_k} = -A_{iq}A_{qi}$$

$$\frac{1}{3}\frac{\partial^2 p}{\partial X_k \partial X_k} = -\frac{A_{iq}A_{qi}}{(\mathbf{C}^{-1})_{ii}}$$

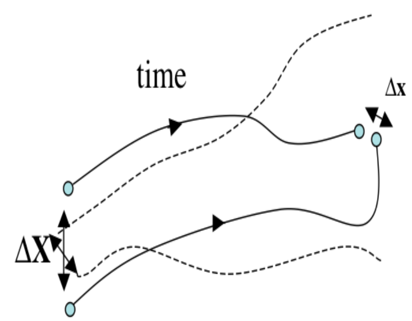
$$\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{(\mathbf{C}^{-1})_{ij}}{(\mathbf{C}^{-1})_{nn}}A_{pq}A_{qp}$$

B. Proposed viscous Hessian model:

Similar approach: Jeong & Girimaji, 2003

$$\nu \frac{\partial^2 \mathbf{A}}{\partial x_i \partial x_j} \approx \frac{\partial X_p}{\partial x_i} \frac{\partial X_q}{\partial x_j} \quad \nu \frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q}$$

$$\nu \frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q} \approx \frac{\nu}{(\delta X)^2} \mathbf{A} \frac{1}{3} \delta_{pq}$$



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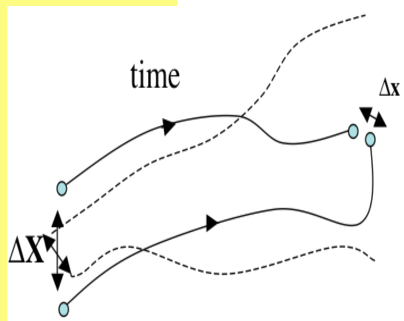
Characteristic Lagrangian displacement during A's Lagrangian correlation time scale (Kolm time):

$$\delta X \sim (disp\ veloc) \times (correl\ time) \sim u' \tau_K$$



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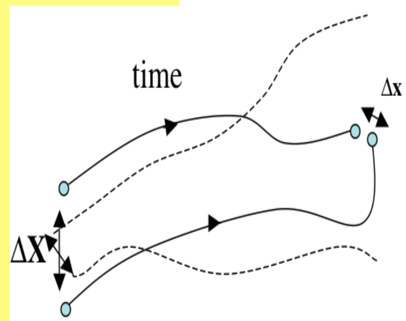
$$\nu \frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q} \approx \frac{\nu}{(\delta X)^2} \mathbf{A} \frac{1}{3} \delta_{pq}$$

Characteristic Lagrangian displacement during A's Lagrangian correlation time scale (Kolm time):

$$\delta X \sim (disp\ veloc) \times (correl\ time) \sim u' \tau_K \sim \lambda$$

$\delta X \sim$ ***Taylor microscale !***

$$(\nu / \delta X^2) \sim \nu / \lambda^2 \sim T^{-1}$$



B. Proposed viscous Hessian model:

Similar approach: Jeong & Girimaji, 2003

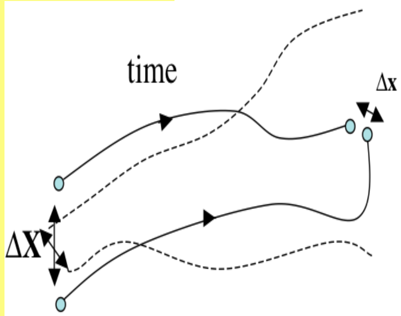
$$\nu \frac{\partial^2 \mathbf{A}}{\partial x_i \partial x_j} \approx \frac{\partial X_p}{\partial x_i}\frac{\partial X_q}{\partial x_j} \; \nu \frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q}$$

$$\nu \frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q} \approx \frac{\nu}{(\delta X)^2}\mathbf{A}\frac{1}{3}\delta_{pq}$$

Characteristic Lagrangian displacement during A's Lagrangian correlation time scale (Kolm time):

$$\delta X \sim (disp\ veloc) \times (correl\ time) \sim u'\tau_K \sim \lambda$$

$\delta X \sim$ ***Taylor microscale !***



$$(\nu / \delta X^2) \sim \nu / \lambda^2 = T^{-1}$$

$$\nu \frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q} \approx \frac{1}{T}\mathbf{A}\frac{1}{3}\delta_{pq}$$

$$\nu \frac{\partial^2 A_{ij}}{\partial x_m \partial x_m} \approx -\frac{(\mathbf{C}^{-1})_{mm}}{3T} A_{ij}$$

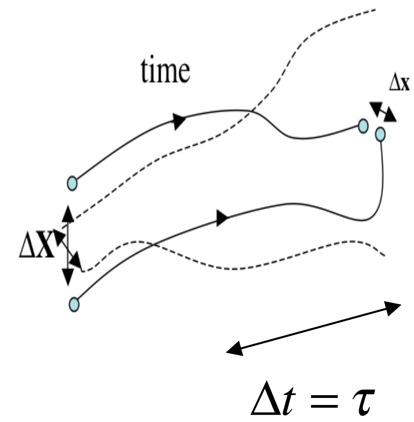
C. Short-time memory material deformation (Markovianization):

Equation for deformation tensor: $D_{ij} = \frac{\partial x_j}{\partial X_i}$ $C_{ij} = D_{ik} D_{jk}$

$$\frac{d\mathbf{D}}{dt} = \mathbf{DA}$$

*Formal Solution in terms of time-ordered **matrix exponential** function ….*

"Markovianiation":
*assume **A** is constant for "some self-correlation time"*



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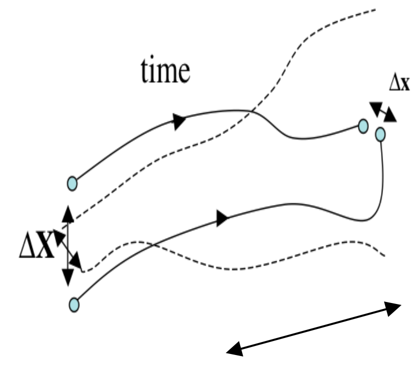

$$\Delta t = \tau$$

"Markovianiation":
assume ***A*** *is constant for "some self-correlation time"*

$$\mathbf{D}(t) = \mathbf{D}(0)\exp(\mathbf{A}\tau)$$

$$C_{ij} = D_{ik} D_{jk}$$

Short-time (Markovian) Cauchy-Green:

$$\mathbf{C}_\tau(t) \approx \exp[\mathbf{A}(t)\tau]\exp[\mathbf{A}^T(t)\tau]$$

The "Recent Fluid Deformation (RFDA) Approximation"

(Chevillard & CM, Phys. Rev. Lett. 2006)

Lagrangian stochastic model for A:

Set of 8 coupled nonlinear stochastic DE's:

$$d\mathbf{A} = \left(-\mathbf{A}^2 + \frac{Tr(\mathbf{A}^2)}{Tr(\mathbf{C}_\tau^{-1})}\mathbf{C}_\tau^{-1} - \frac{Tr(\mathbf{C}_\tau^{-1})}{3T}\mathbf{A} \right) dt + d\mathbf{W}$$

$$\mathbf{C}_\tau(t) = \exp[\mathbf{A}(t)\tau_K]\exp[\mathbf{A}^T(t)\tau_K]$$

Parameter: Reynolds number

$$\frac{\tau_K}{T} = c\,\mathrm{Re}^{-1/2}$$

dW: white-in-time Gaussian forcing (trace-free-isotropic-covariance structure - unit variance (in units of T)

Results:

Simulate SDE
and sample statistics

$$\frac{\tau_K}{T} = 0.1$$

$$d\mathbf{A} = \left(-\mathbf{A}^2 + \frac{Tr(\mathbf{A}^2)}{Tr(\mathbf{C}_\tau^{-1})}\mathbf{C}_\tau^{-1} - \frac{Tr(\mathbf{C}_\tau^{-1})}{3T}\mathbf{A} \right) dt + d\mathbf{W}$$

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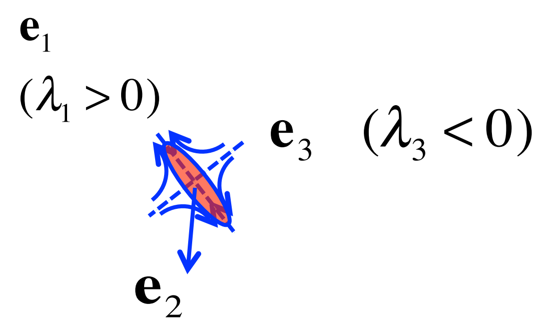

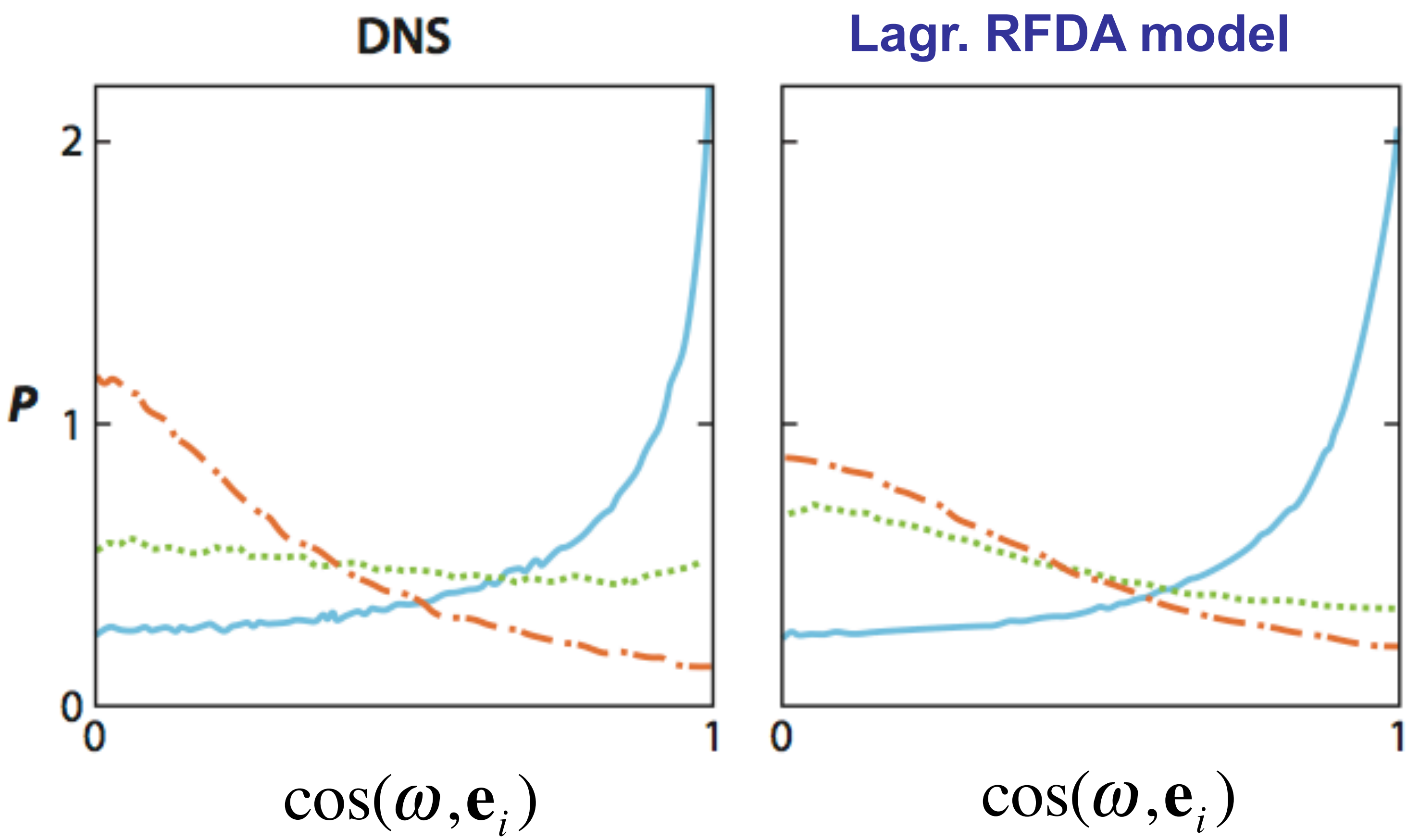

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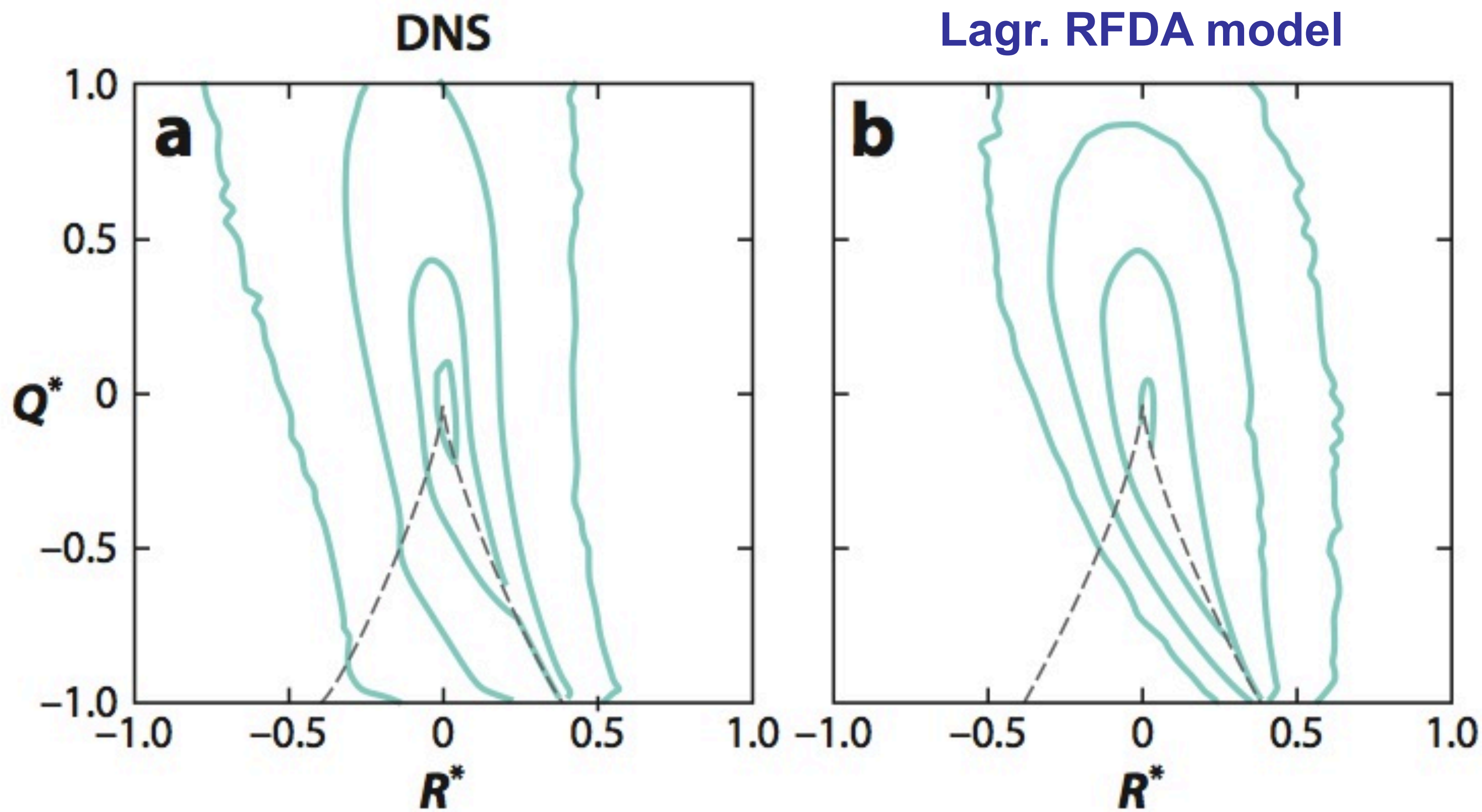

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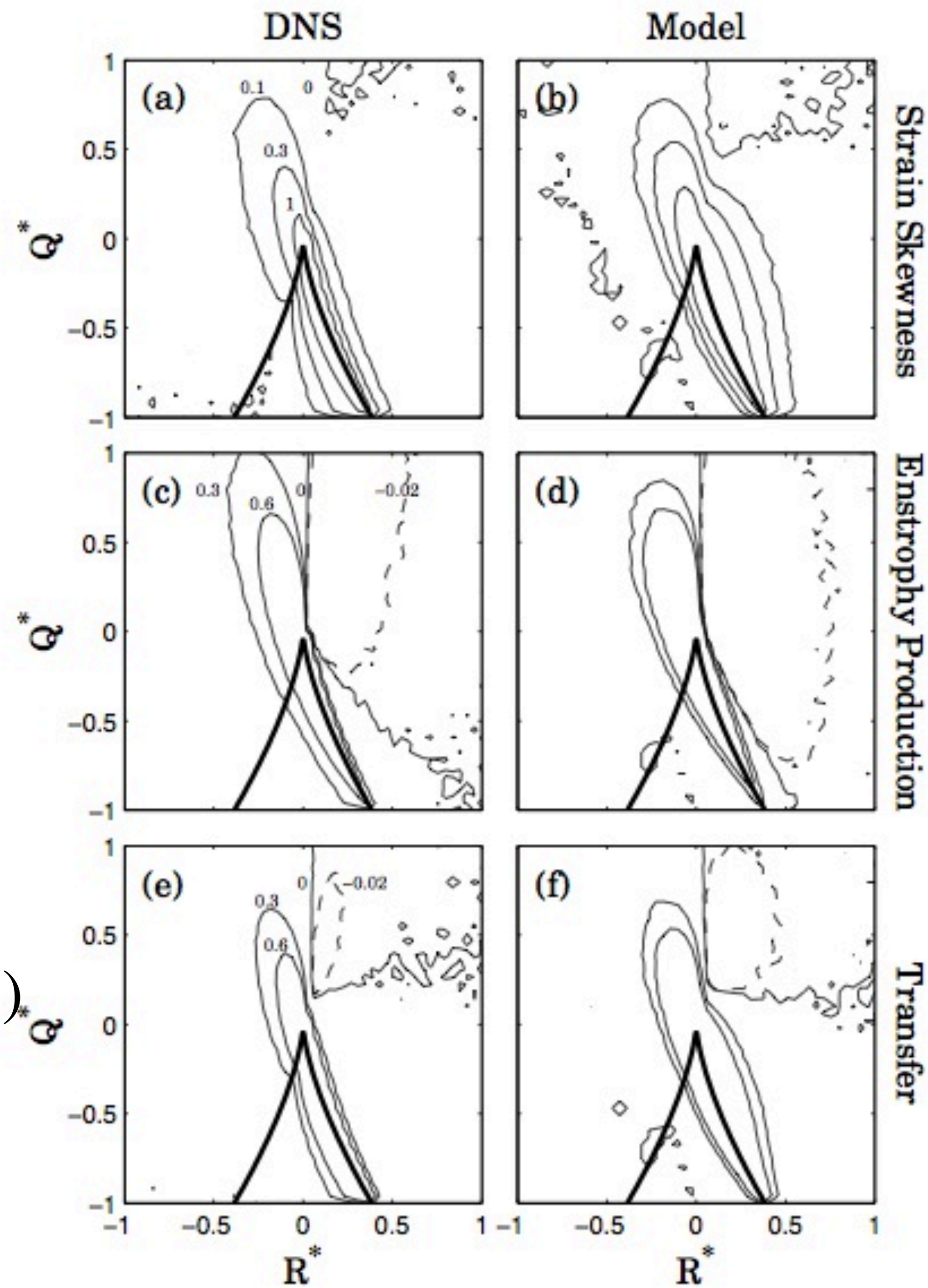
$$\mathbf{C}_\tau(t) \approx \exp[\mathbf{A}(t)\tau]\exp[\mathbf{A}^T(t)\tau]$$

$$-\langle \mathrm{Tr}(\mathbf{S}^3) \,|\, Q^*, R^* \rangle \mathcal{P}(Q^*, R^*)$$

$$\langle \omega_i S_{ij} \omega_j \,|\, Q^*, R^* \rangle \mathcal{P}(Q^*, R^*)$$

$$-\langle \mathrm{Tr}(\mathbf{A}^2 \mathbf{A}^T) \,|\, Q, R \rangle \mathcal{P}(Q^*, R^*)$$



Chevillard, CM, Biferale,
Toschi (PoF 2008)

Results:

Simulate SDE
and sample statistics

$$d\mathbf{A} = \left(-\mathbf{A}^2 + \frac{Tr(\mathbf{A}^2)}{Tr(\mathbf{C}_\tau^{-1})}\mathbf{C}_\tau^{-1} - \frac{Tr(\mathbf{C}_\tau^{-1})}{3T}\mathbf{A} \right) dt + d\mathbf{W}$$

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Chevillard, CM, Biferale, Toschi (PoF 2008)

$$\frac{\tau_K}{T} = 0.1$$

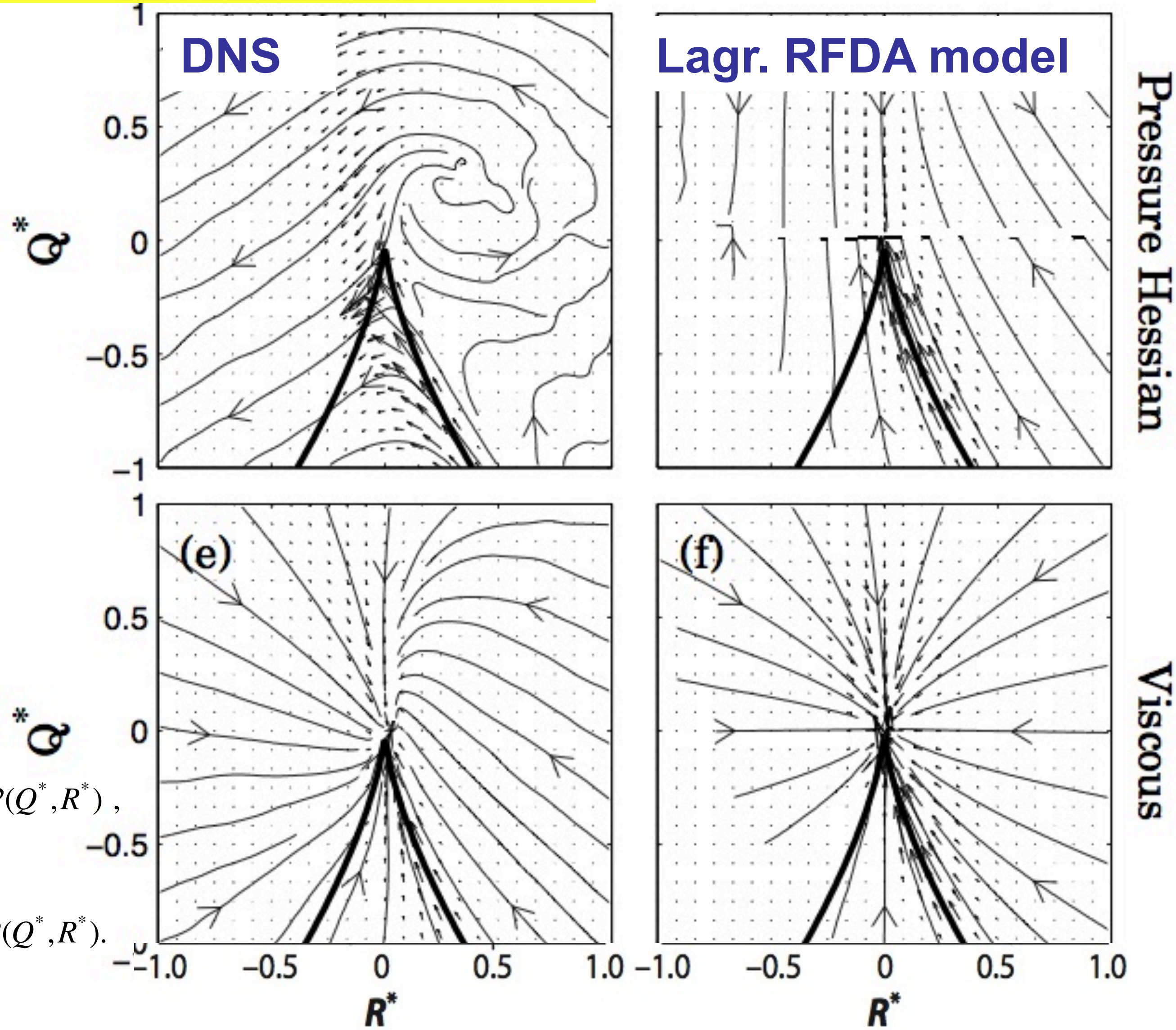
Fokker-Planck

$$\frac{\partial \mathcal{P}}{\partial t^*} + \begin{pmatrix} \dfrac{\partial}{\partial Q^*} \\ \dfrac{\partial}{\partial R^*} \end{pmatrix} . \vec{\mathcal{W}} = 0 \ ,$$

Probability currents:

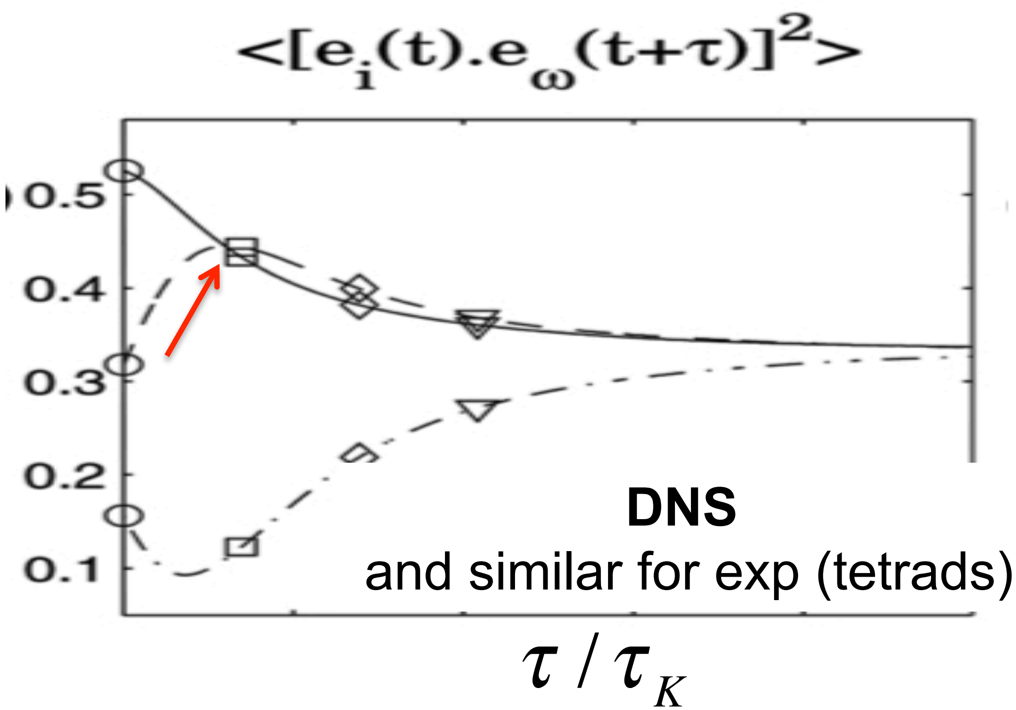
$$\vec{\mathcal{W}}_p = \left\langle \begin{pmatrix} -A_{ik}H^p_{ki}/\sigma^3 \\ -A_{ik}A_{kl}H^p_{li}/\sigma^4 \end{pmatrix} \middle| Q^*, R^* \right\rangle \mathcal{P}(Q^*, R^*) \ ,$$

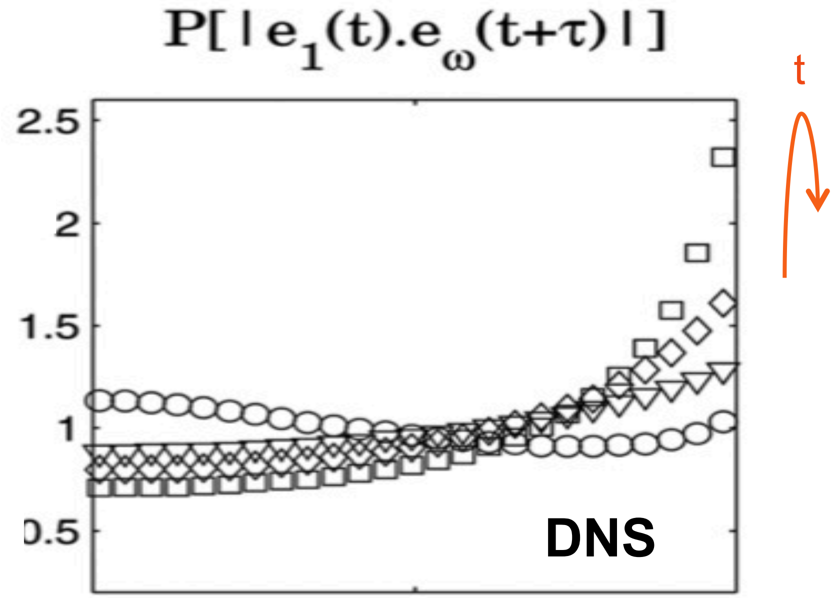
$$\vec{\mathcal{W}}_v = \left\langle \begin{pmatrix} -A_{ik}H^v_{ki}/\sigma^3 \\ -A_{ik}A_{kl}H^v_{li}/\sigma^4 \end{pmatrix} \middle| Q^*, R^* \right\rangle \mathcal{P}(Q^*, R^*) .$$

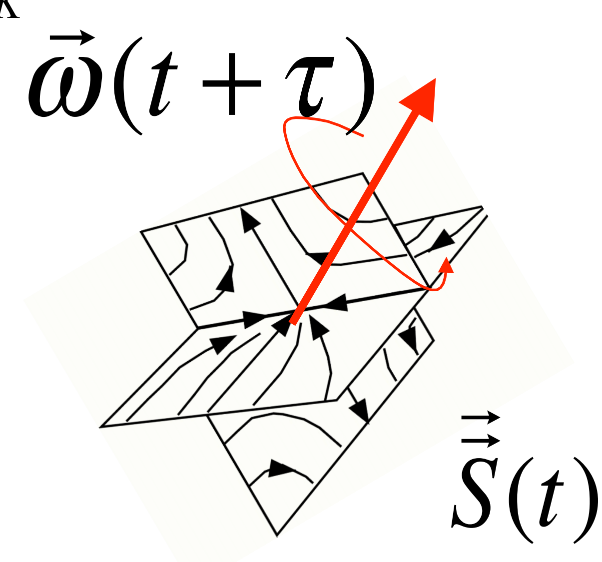


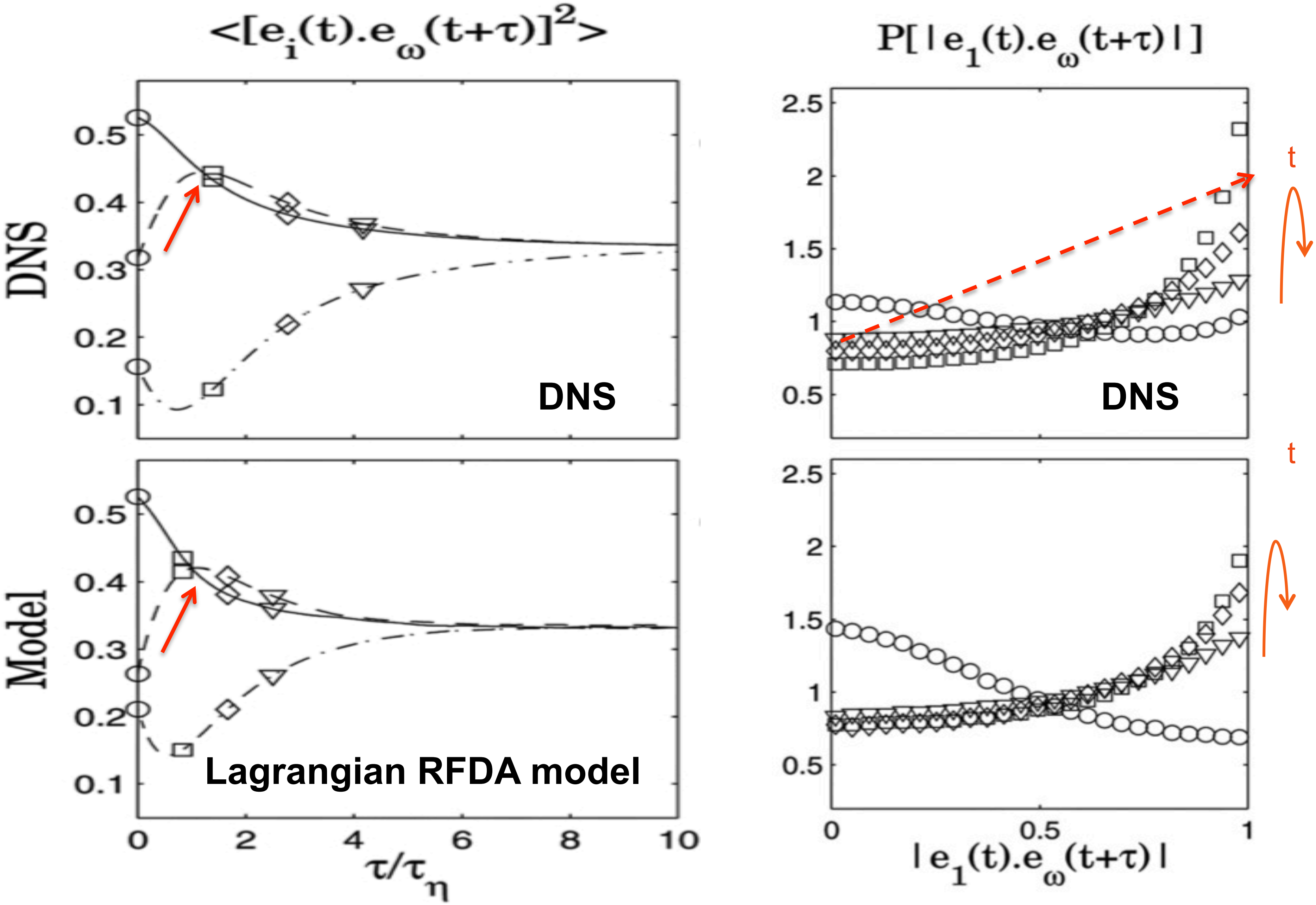
H. Xu, A. Pumir, E. Bodenschatz (2011) Nature Physics 7, 709

see also preprint (Pumir et al. 2012, arxiv, Bodenschatz discussion)









<[e_i(t).e_ω(t+τ)]^2>
P[|e_1(t).e_ω(t+τ)|]
DNS
Model
DNS
DNS
Lagrangian RFDA model
t
t
τ/τ_η
|e_1(t).e_ω(t+τ)|

Predictions of anomalous scaling,

Power-laws of velocity gradients as function of Reynolds number $(\tau/T)^2$??

Running the model at arbitrarily high Re?

$$d\mathbf{A} = \left(-\mathbf{A}^2 + \frac{Tr(\mathbf{A}^2)}{Tr(\mathbf{C}_\tau^{-1})}\mathbf{C}_\tau^{-1} - \frac{Tr(\mathbf{C}_\tau^{-1})}{3T}\mathbf{A} \right) dt + d\mathbf{W}$$

$$\mathbf{C}_\tau(t) \approx \exp[\mathbf{A}(t)\tau]\exp[\mathbf{A}^T(t)\tau]$$

At different τ/T

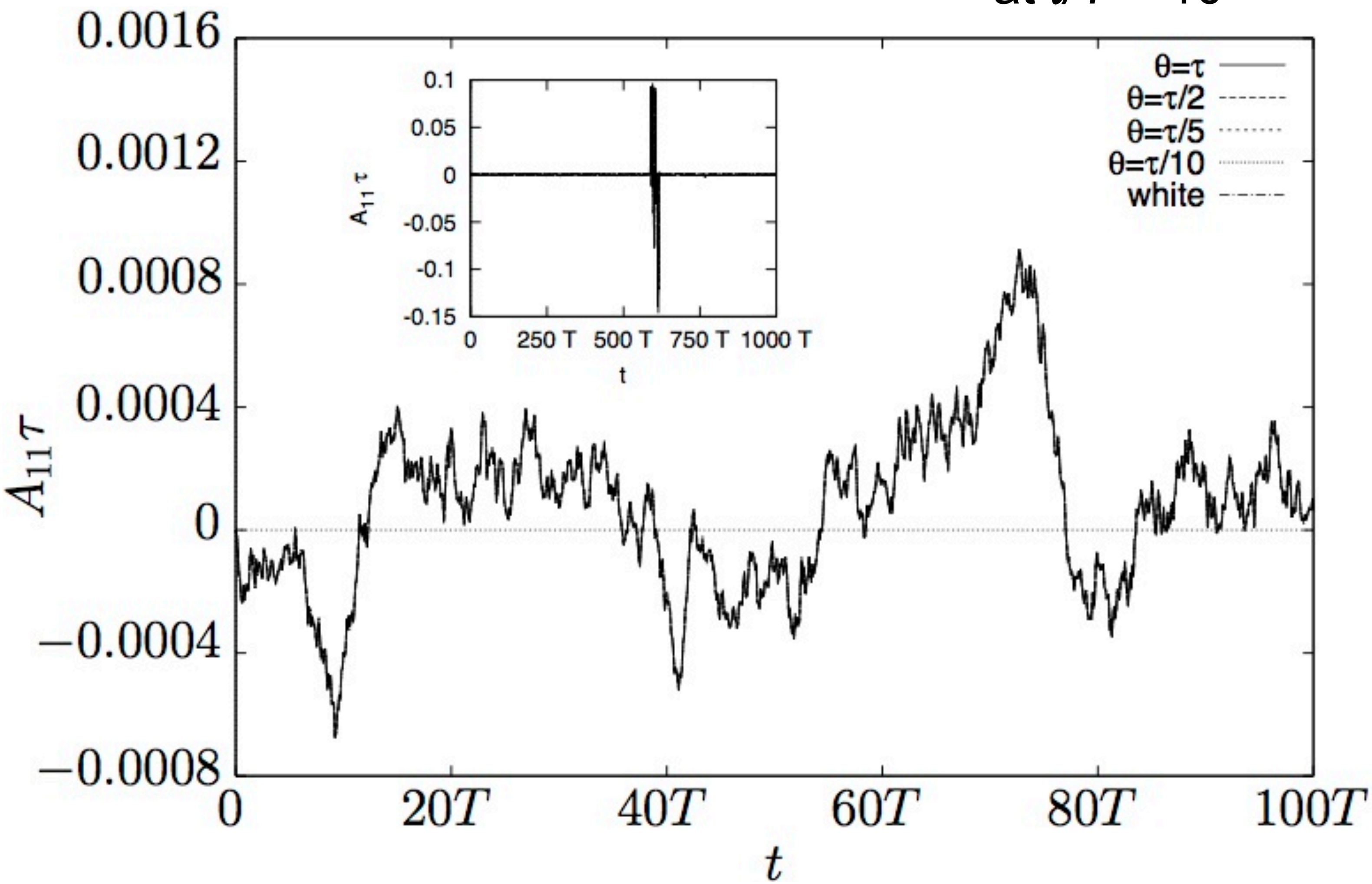
Problems at increasing Reynolds numbers running at $\tau/T = 10^{-3}$:

Martins-Afonso & CM (Physica D 2010)

$$d\mathbf{A} = \left(-\mathbf{A}^2 + \frac{Tr(\mathbf{A}^2)}{Tr(\mathbf{C}_\tau^{-1})}\mathbf{C}_\tau^{-1} - \frac{Tr(\mathbf{C}_\tau^{-1})}{3T}\mathbf{A} \right) dt + d\mathbf{W}$$

$$\mathbf{C}_\tau(t) \approx \exp[\mathbf{A}(t)\tau]\exp[\mathbf{A}^T(t)\tau]$$

at $\tau/T = 10^{-3}$



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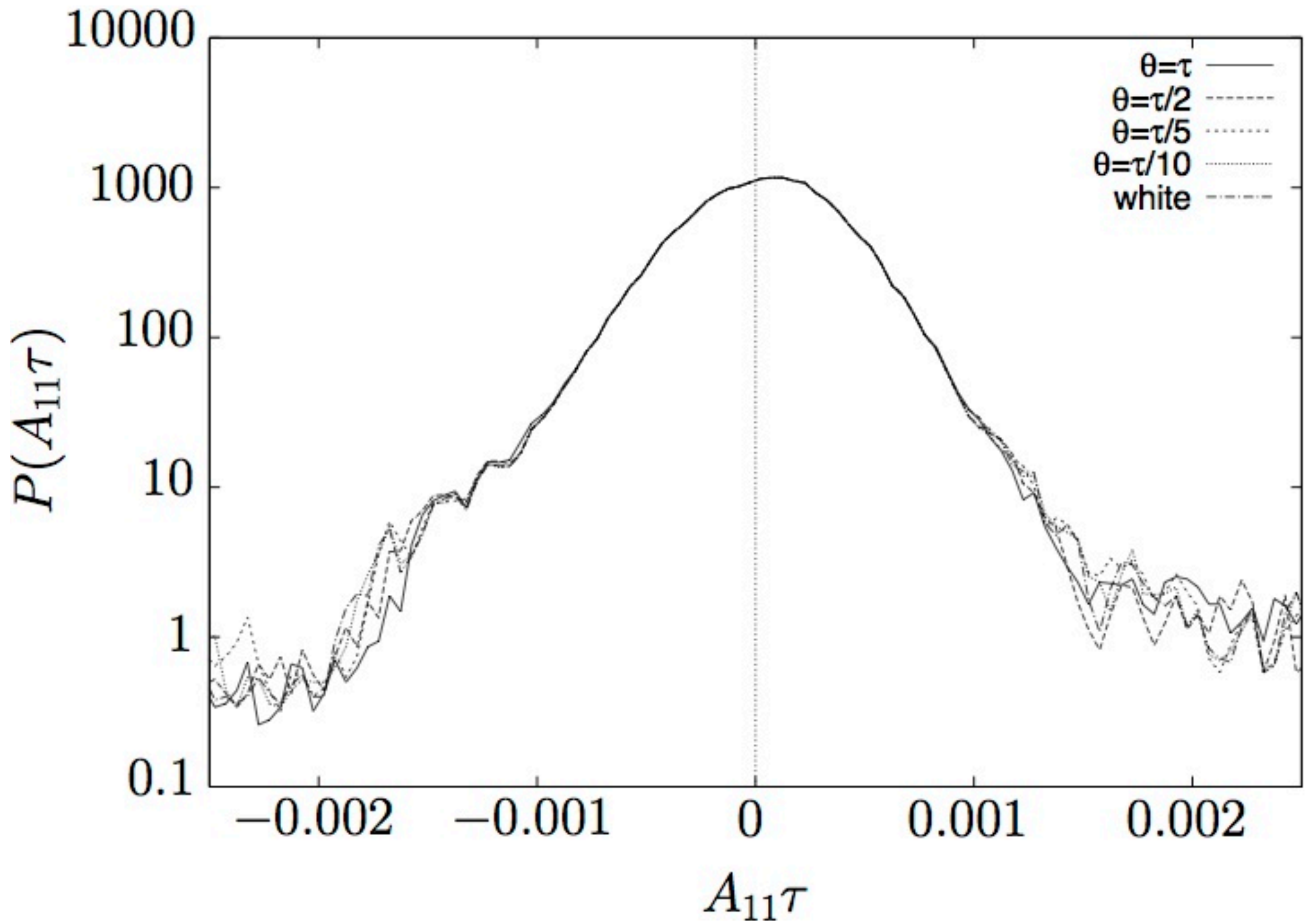
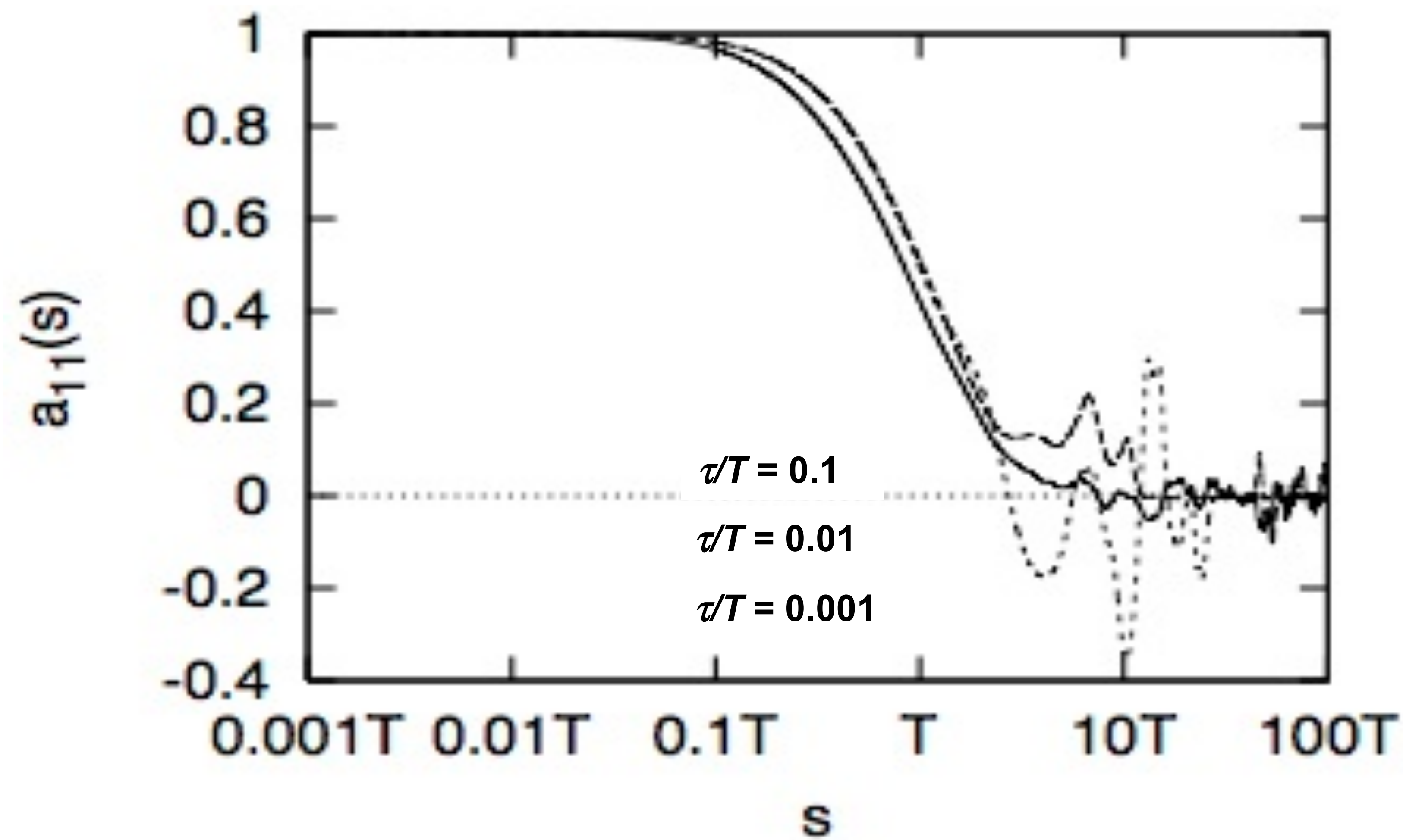


Problems at increasing Reynolds numbers running at various τ/T :

Martins-Afonso & CM (Physica D 2010)

$$a_{ij}(s) \equiv \frac{\langle A_{ij}(t)A_{ij}(t+s)\rangle - \langle A_{ij}(t)\rangle^2}{\langle A_{ij}^2(t)\rangle - \langle A_{ij}(t)\rangle^2}$$



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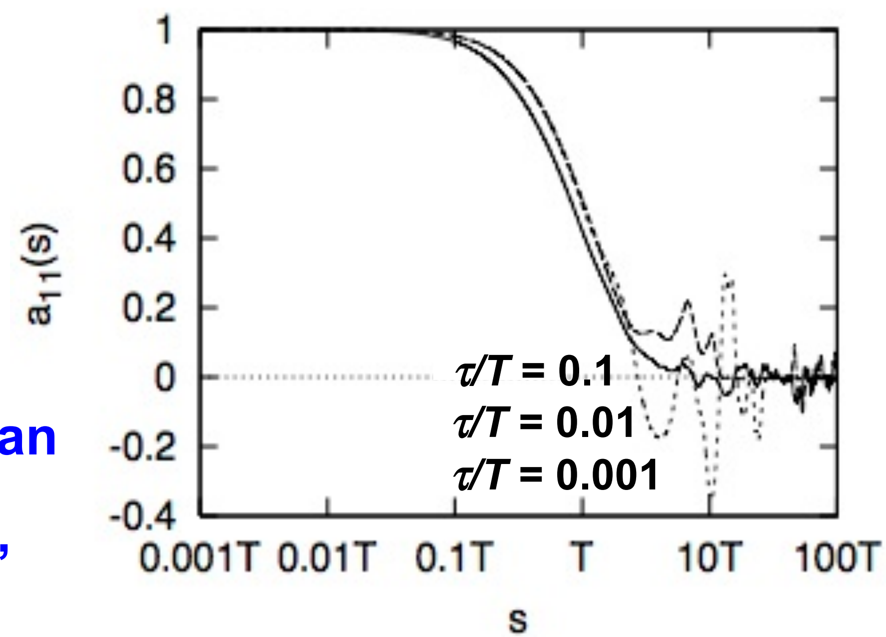
- **Autocorrelation function does not decay at scale τ as assumed in RFDA model**
- **Increase in forcing W no cure, unless so strong that back to Gaussian**
- **RE nonlinearity not "chaotic enough"**

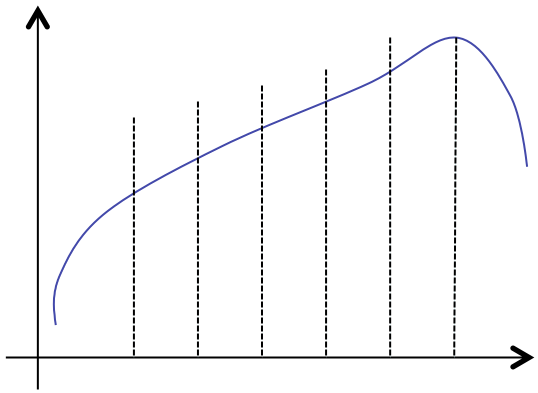


An attempt to include additional degrees of freedom (multi-scale gradients)

The matrix shell model (Biferale, Chevillard, CM & Toschi, PRL 2007)

$$\mathbf{A} = \sum_n \mathbf{A}_n$$

$$\frac{d\mathbf{A}_n}{dt} = -\sum_{p,q} \left(\mathbf{A}_p \mathbf{A}_q\right)_n + \left(\nabla\nabla p\right)_n + \nu \partial^2 \mathbf{A}_n$$



$$k_n = 2^n k_0$$

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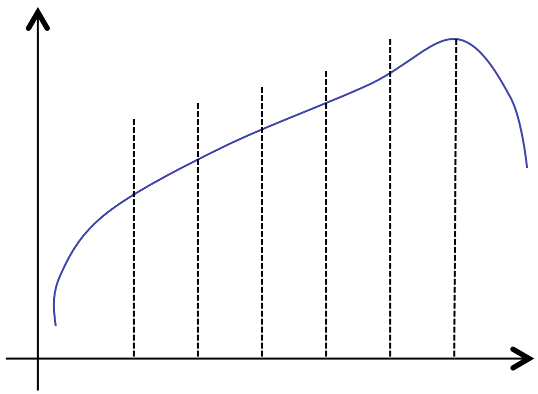

$$k_n = 2^n k_0$$

"Mixed" Restricted Euler – shell model dynamics:

$$\frac{d\mathbf{A}_n}{dt} = \alpha\left(-\mathbf{A}_n^2 + \frac{1}{3}\mathrm{Tr}(\mathbf{A}_n^2)\mathbf{I}\right) + (1-\alpha)\left(\mathbf{F}_n^d - \nu k_n^2\mathbf{A}_n\right)$$

$$\mathbf{F}_n = \mathbf{A}_{n+2}\mathbf{A}_{n+1}^T + b2^2\mathbf{A}_{n-1}^T\mathbf{A}_{n+1} + (1-b)2^4\mathbf{A}_{n-2}\mathbf{A}_{n-1}$$

$$E = \sum_n k_n^{-2}\mathrm{Tr}\left(\mathbf{A}_n\mathbf{A}_n^T\right)$$ is conserved by **F** term

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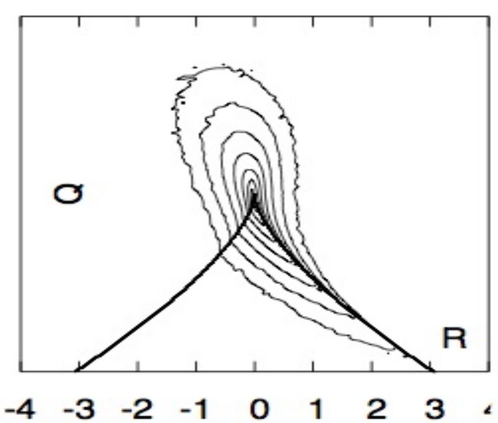
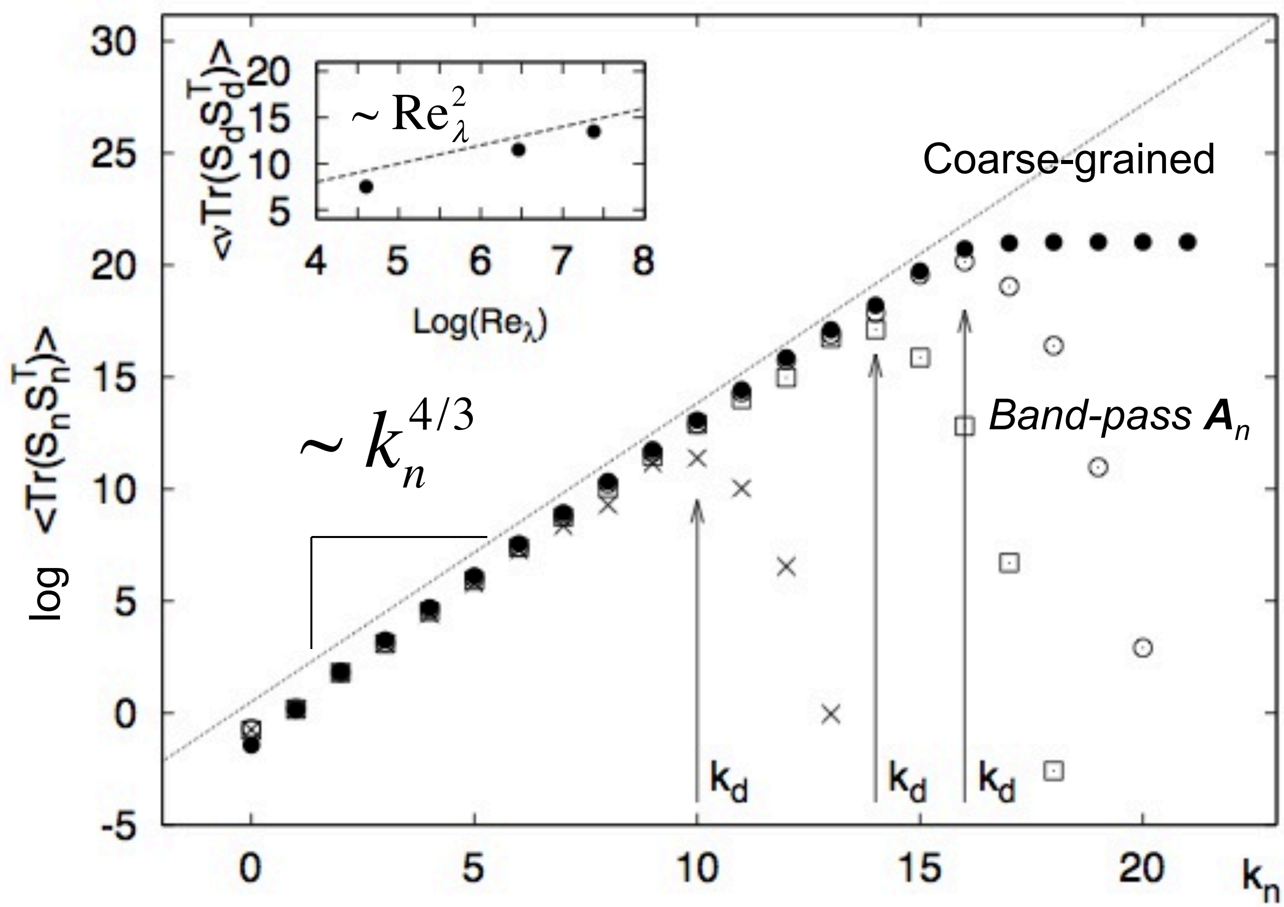
$\alpha = 0.5 \qquad b = 0.5$

Results

$N = 14,\ 18,\ 22$

$\mathrm{Re}_\lambda = 130,\ 640,\ 1500$





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$\alpha = 0.5 \qquad b = 0.5$

$N = 14,\ 18,\ 22$

$Re_\lambda = 130,\ 640,\ 1500$

Flatness of A_{11}

PDFs of A_{11} and A_{12}

(long + transverse)

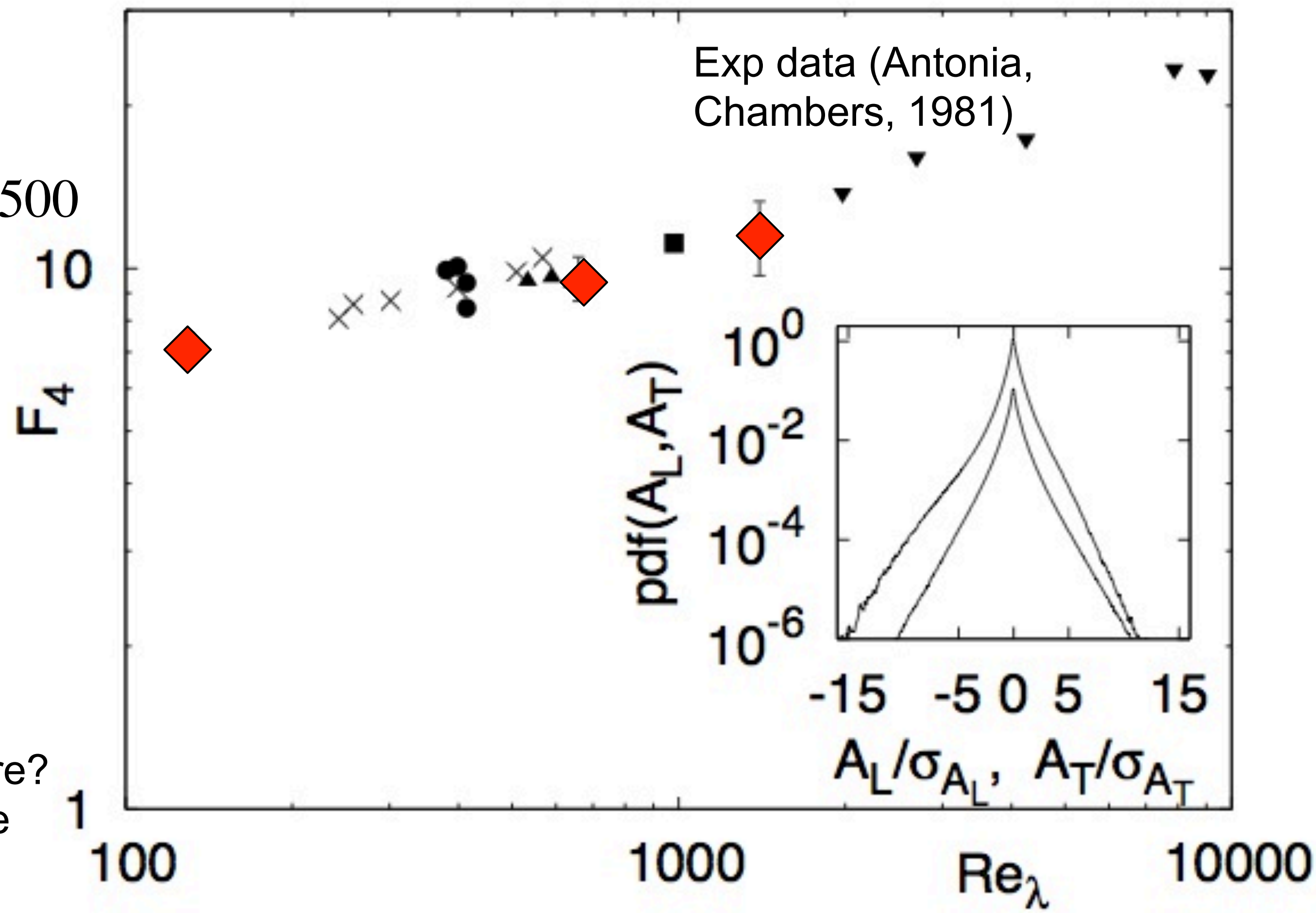
Skewness of A_{11}

Tails

Physical basis for closure?

(no recourse to pressure

Hessian physics …)



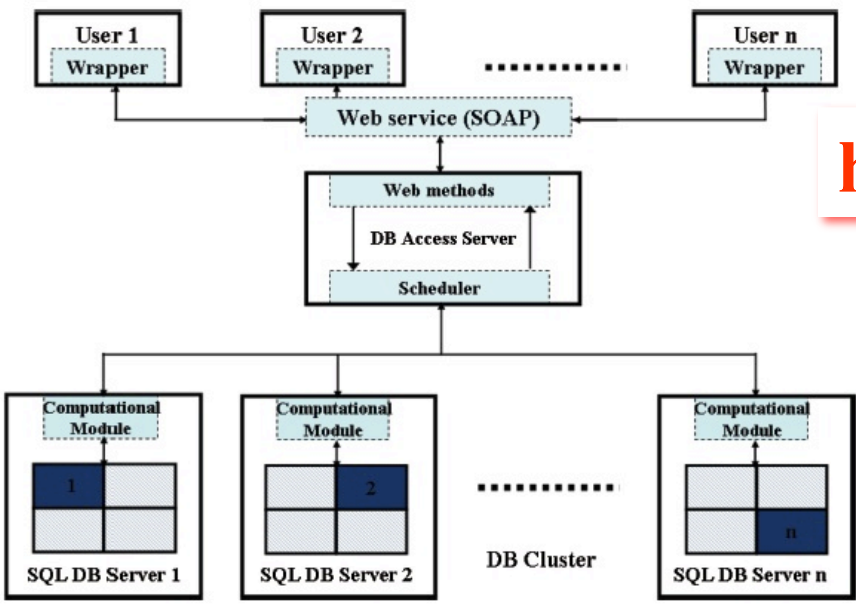
Next: some hopefully useful observations
(but that have not led yet to any new model)

- Correlations between “real” pressure Hessian and model
- Structure of pressure Hessian

(Use data from JHU public database)

Take 1024^3 DNS of forced isotropic turbulence

(standard pseudo-spectral Navier-Stokes simulation, dealiased):



1024^4 space-time history

27 Tbytes, $Re_l \sim 430$

http://turbulence.pha.jhu.edu

Y. Li, E. Perlman, M. Wan, Y. Yang, R. Burns, C.M., R. Burns, S. Chen, A. Szalay & G. Eyink:

"A public turbulence database cluster and applications to study Lagrangian evolution of velocity increments in turbulence".

***Journal of Turbulence* 9,** No 31, 2008.

So far 12 papers published using data from public database

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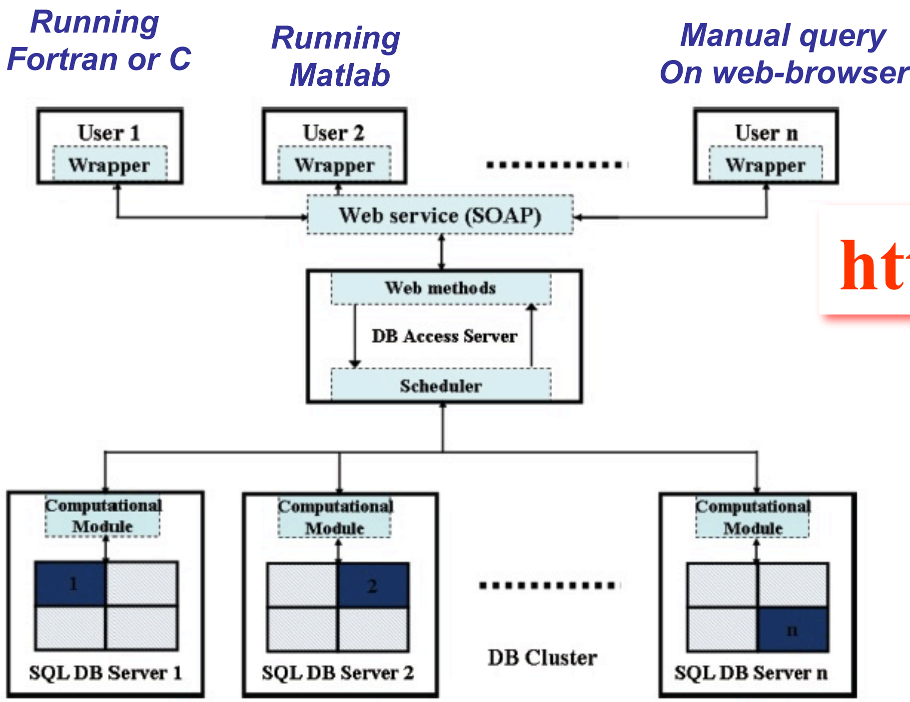

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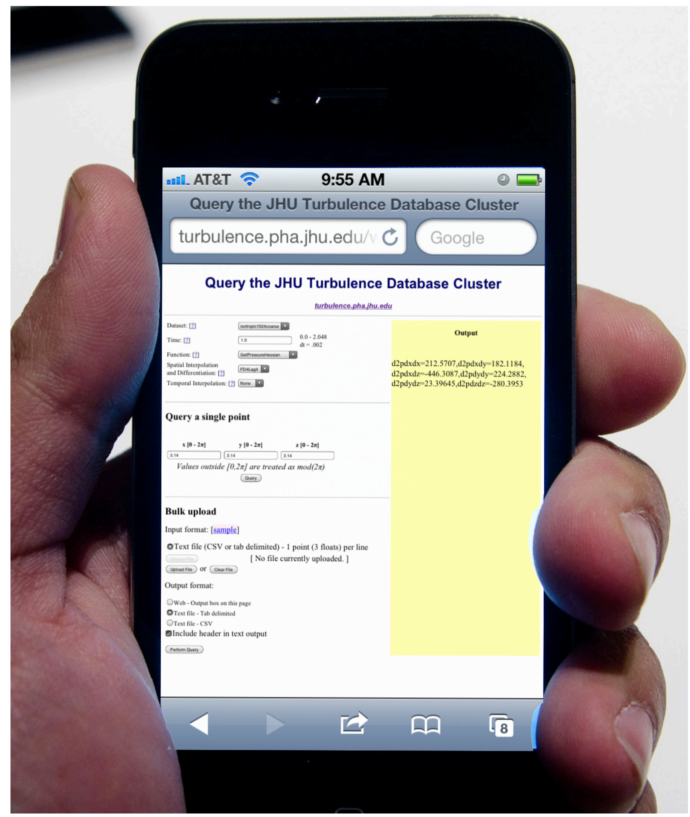

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***Journal of Turbulence* 9,** No 31, 2008.

So far 12 papers published using data from public database

New paradigm:

Client computer (e.g. my laptop) runs the analysis using Fortran, C, Matlab codes, fetching data as needed from databases through a web-service

we adapted Fortran, C and Matlab to "surf the web" for data

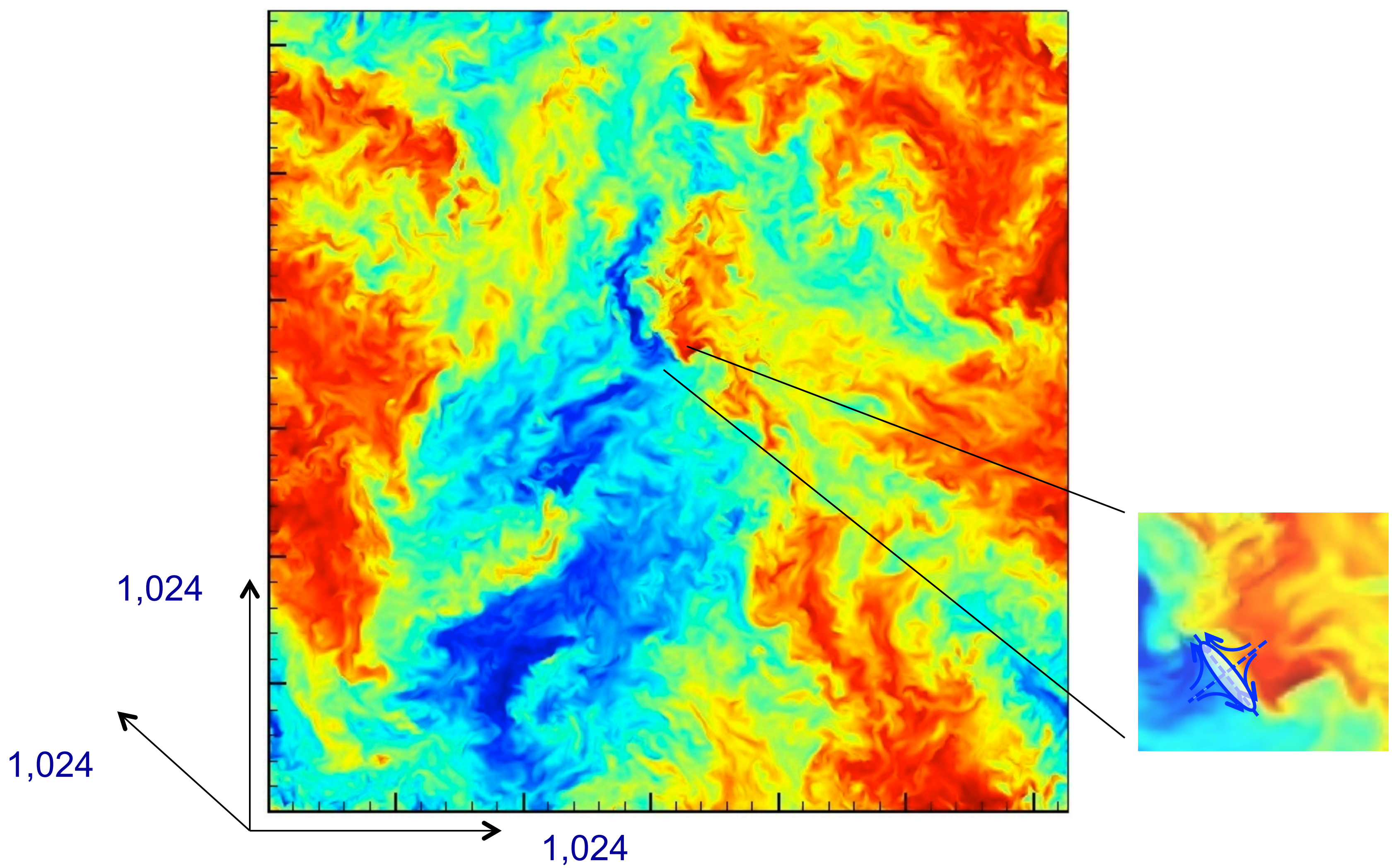
Y. Li, E. Perlman, M. Wan, Y. Yang, R. Burns, C. Meneveau, R. Burns, S. Chen, A. Szalay & G. Eyink: ***Journal of Turbulence* 9,** No 31, 2008.

```
  ! This is required before any WebService routines are called.
  !
  CALL soapinit()

  ! Enable exit on error.  See README for details.
  CALL turblibSetExitOnError(1)

  dx = 2*3.1415926535/1024.
  ind = 0
  do i = 1, 1024
   do j = 1, 1024
    ind = ind+1
    points(1, ind) = i*dx
    points(2, ind) = j*dx
    points(3, ind) = 3.1415926535
  end do
  end do
  write(*,*)
  write(*,*) 'Requesting velocity at 1024x1024 points...'

  rc = getvelocity(authkey, dataset, 1.00, 0, 0, 1048576, points, dataout3)

  ind=0
  do i = 1, 1024
   do j = 1, 1024
    ind = ind+1
    u(i,j) = dataout3(1,ind)
  end do
  end do
  !
  ! Destroy the gSOAP runtime.
  ! No more WebService routines may be called.
  !
  CALL soapdestroy()

end program TurbTest

CMMacBookPro-2:turblib-20111031 meneveau$
```

1,024^3 DNS: iso-velocity filled contours (R_λ=433)

(data from: JHU public database cluster, Claire Verhulst & Jason Graham Matlab visualization)

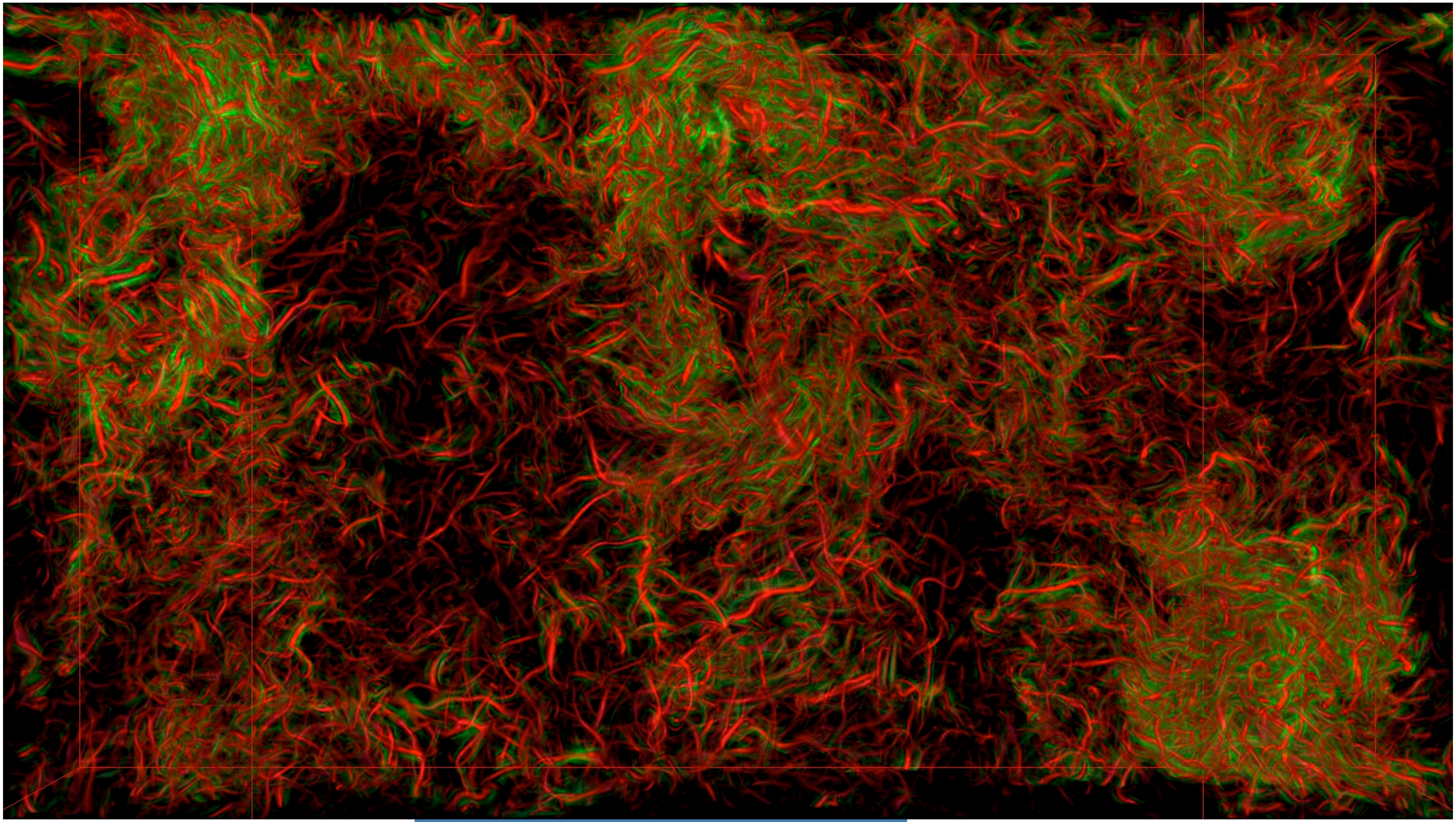
$$u_1(x, y, z_0, t_0)$$



iso-vorticity surfaces

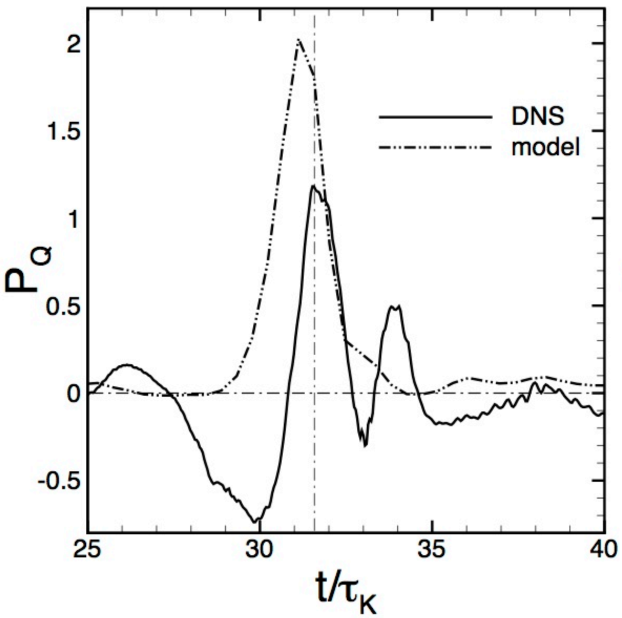
(JHU database, Dr. Kai Buerger visualization)

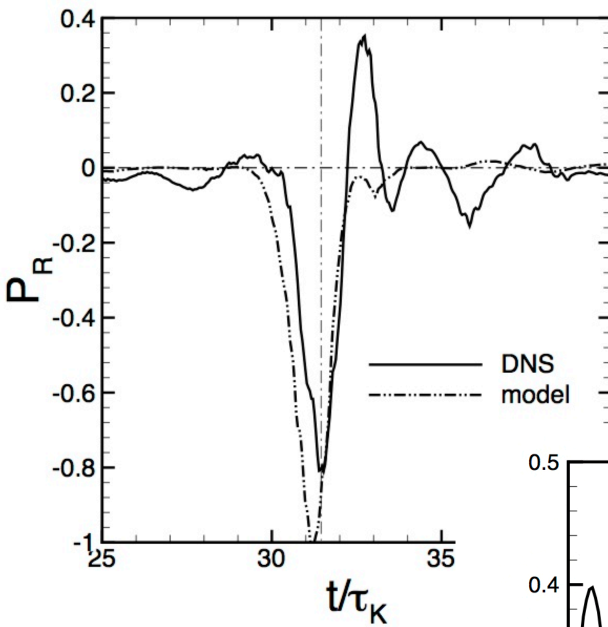
$$\left\|(\nabla \times \mathbf{u})^2\right\|$$



Comparing pressure Hessian tensor invariants

RFD model & database, Lagrangian time series





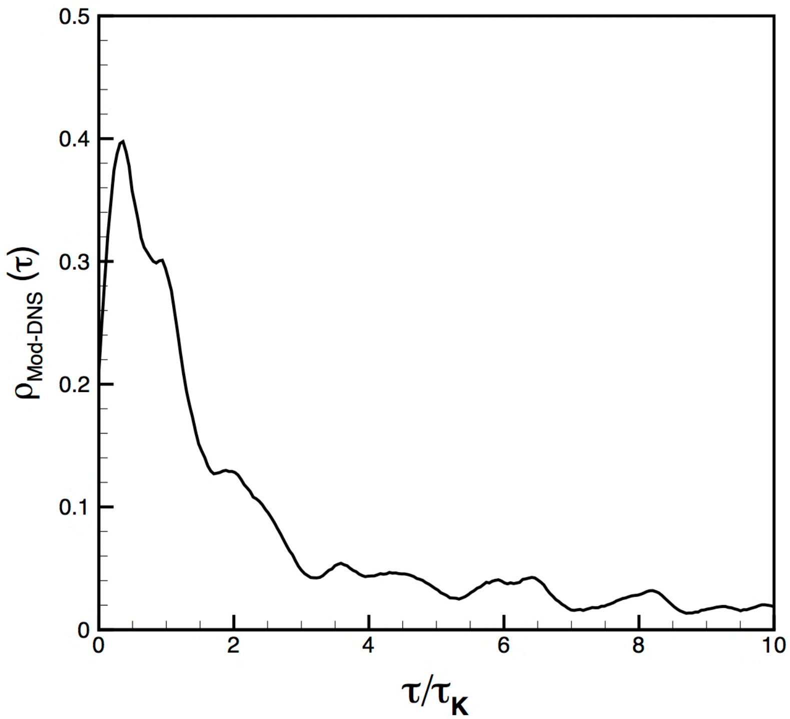
$$P_Q = -A_{ik} H^p_{ki}$$

$$P_R = -A_{ik} A_{kl} H^p_{li}$$

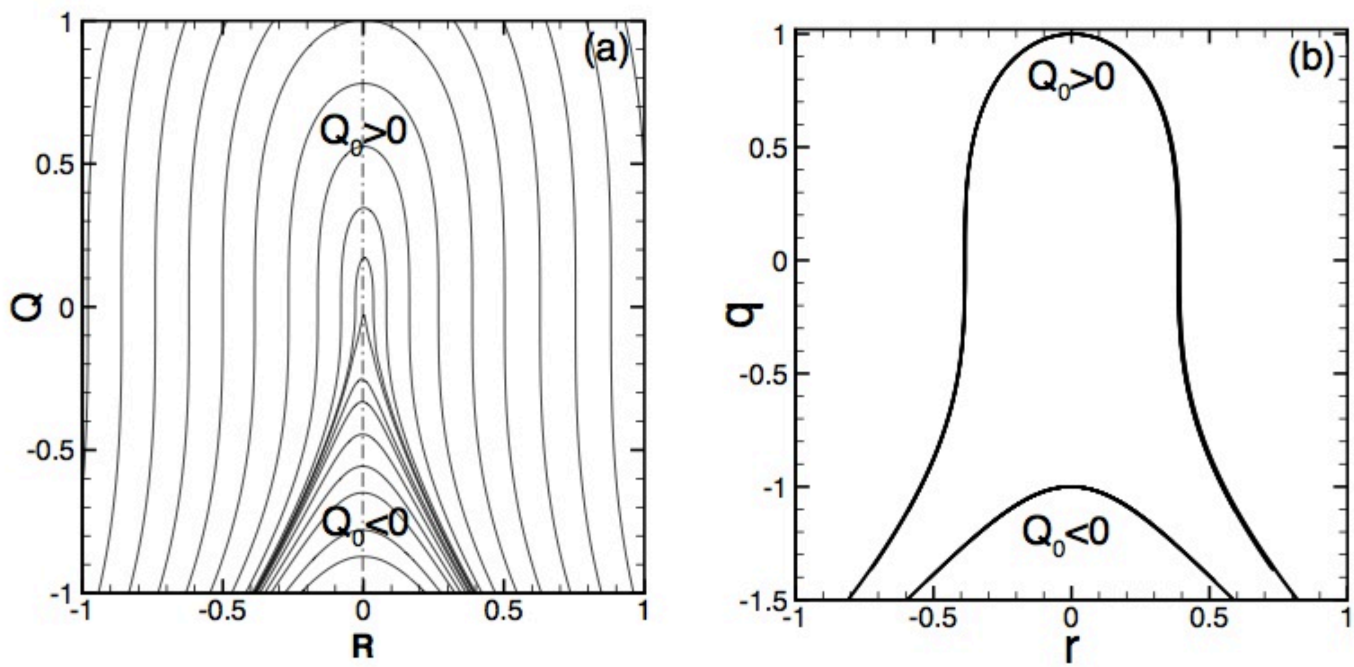
Yu et al. (2012), J. of Turbulence **13**, N12.

Local “closure” comes in “too early” (overestimates immediacy). Time – delay ~ 0.5 τ_K.

About 40% correlation only



Conditional averages, probability current, another look:



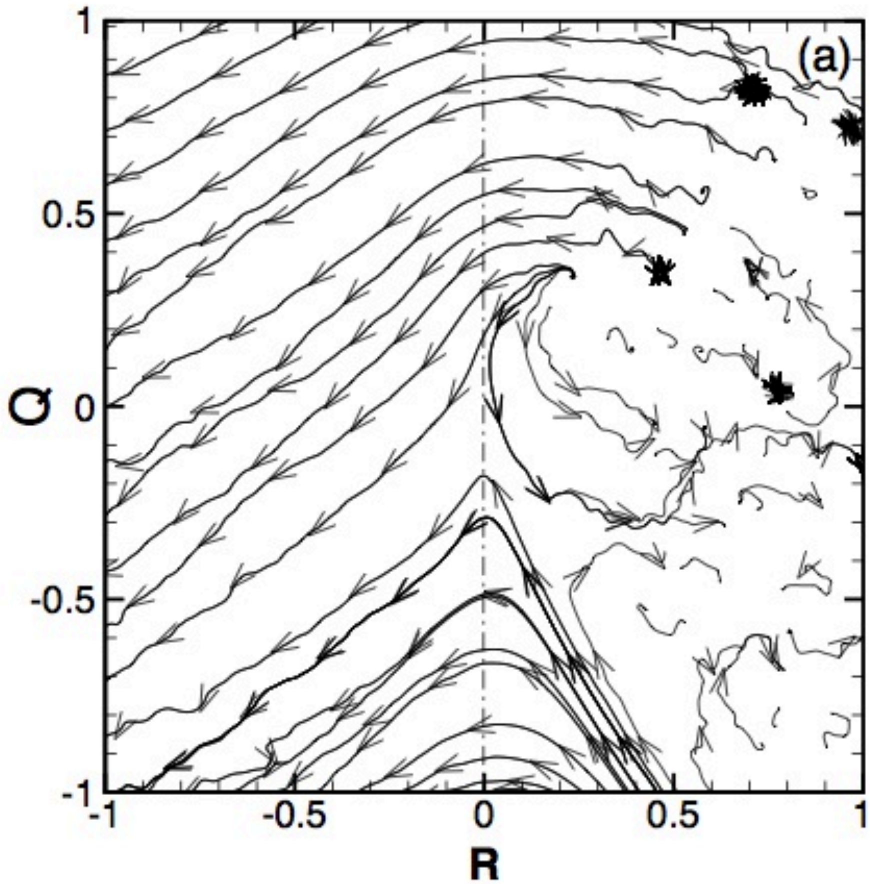
$$r = \frac{R}{|Q_0|^{3/2}}, \qquad q = \frac{Q}{|Q_0|}, \qquad Q_0^3 = Q^3 + \frac{27}{4}R^2$$

$$q = \left(sign(Q_0) - \frac{27}{4} r^2 \right)^{3/2}$$

Is there such a "collapse" for pressure Hessian?

Conditional averages, probability current, another look:

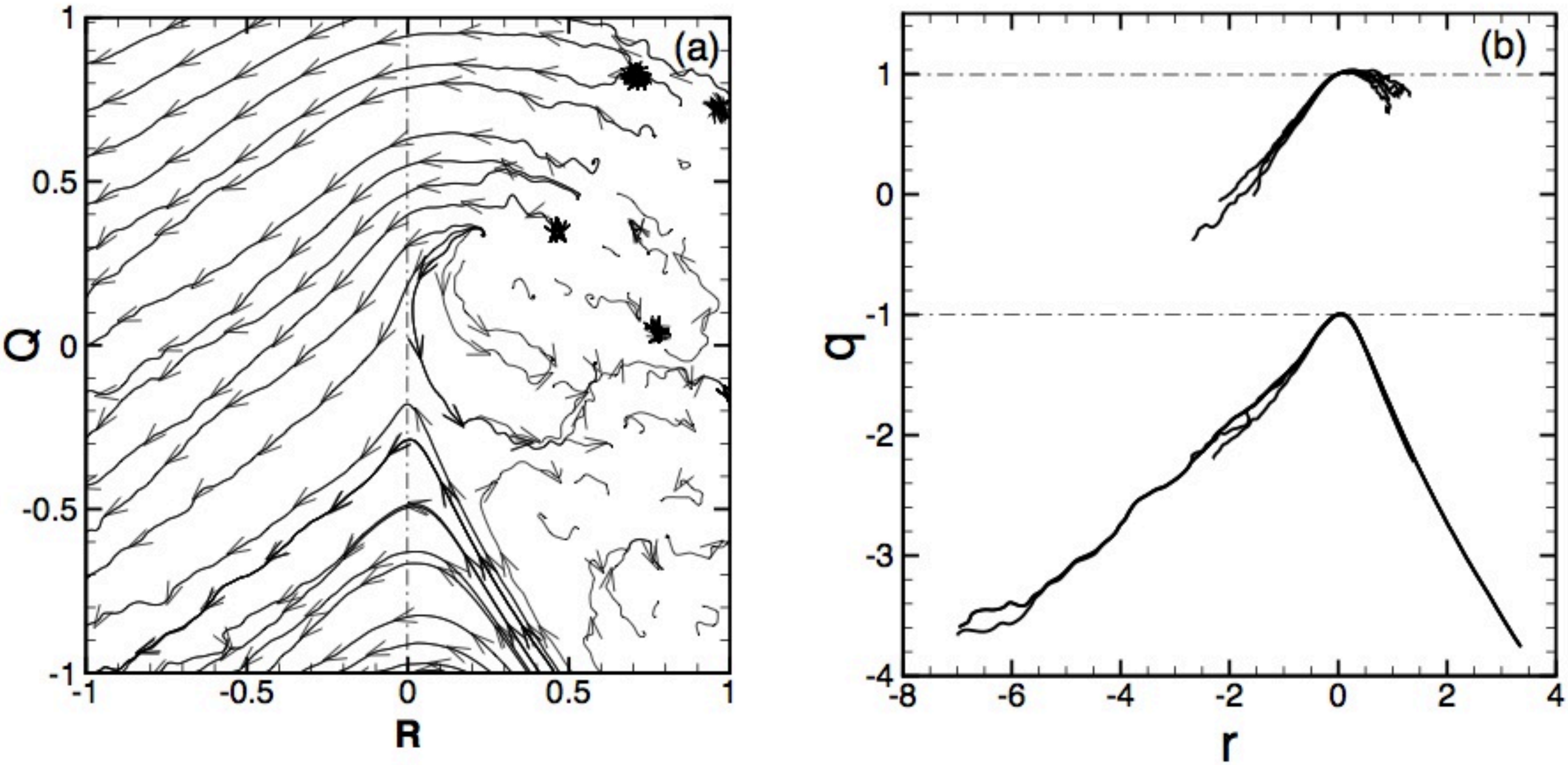
Yu & CM (2012), in preparation.



$$\vec{\mathcal{W}}_p = \left\langle \begin{pmatrix} -A_{ik}H^p_{ki}/\sigma^3 \\ -A_{ik}A_{kl}H^p_{li}/\sigma^4 \end{pmatrix} \Big| Q^*, R^* \right\rangle \mathcal{P}(Q^*, R^*)$$

Conditional averages, probability current, another look:

Yu & CM (2012), in preparation.



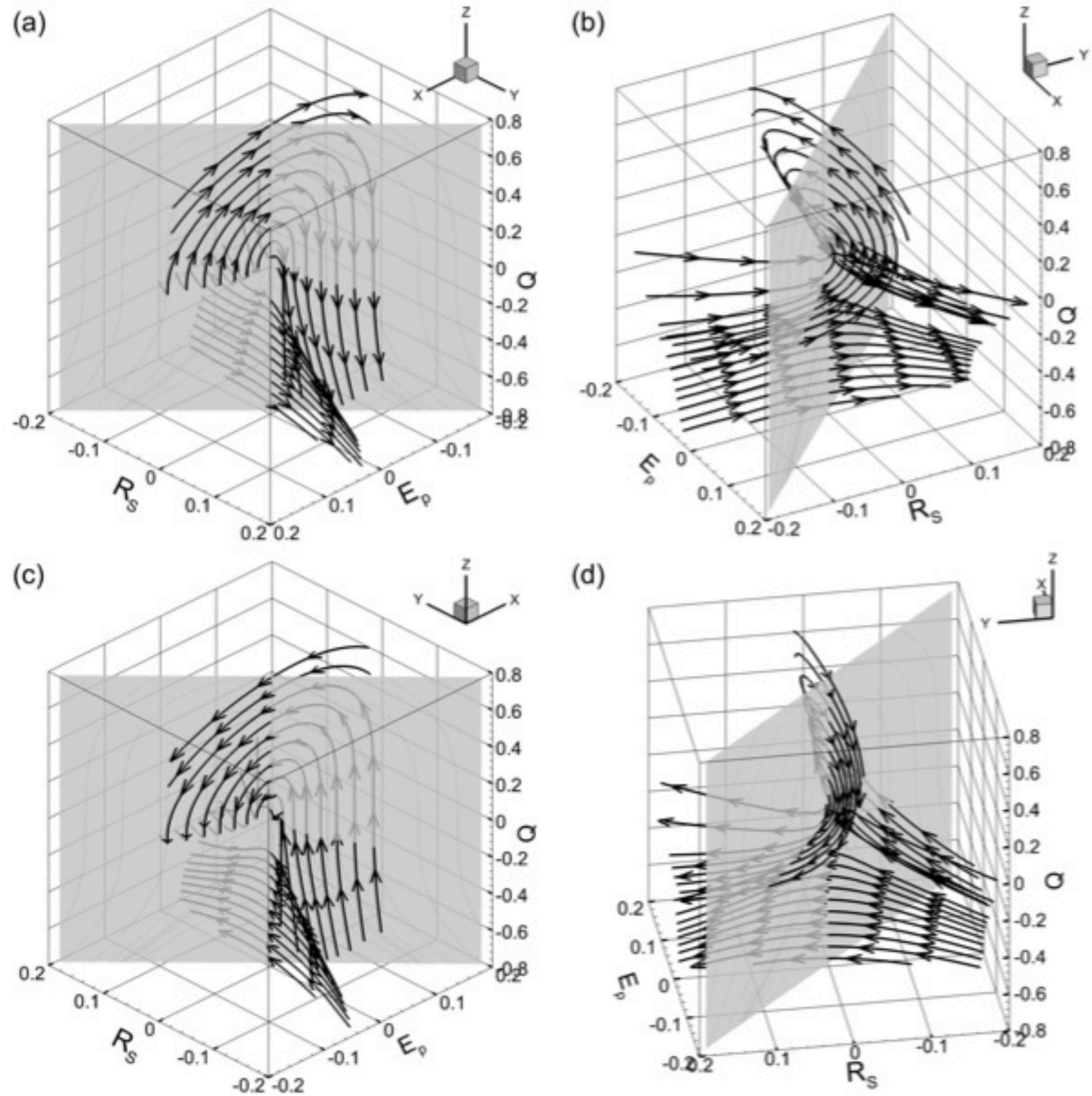
$$\vec{\mathcal{W}}_p = \left\langle \begin{pmatrix} -A_{ik}H^{p}_{ki}/\sigma^3 \\ -A_{ik}A_{kl}H^{p}_{li}/\sigma^4 \end{pmatrix} \Big| Q^*, R^* \right\rangle \mathcal{P}(Q^*, R^*)$$

$$q = \frac{Q}{Q_{P0}} \qquad r = \frac{R}{Q_{PO}^{3/2}}$$

Same, but in 3D expanded RQ-diagram:

Yu & CM (2012), in preparation.

B. Luethi, M. Holzner, & A. Tsinober, "Expanding the Q-R space to three dimensions", J. Fluid Mech. 641, 497 (2010)



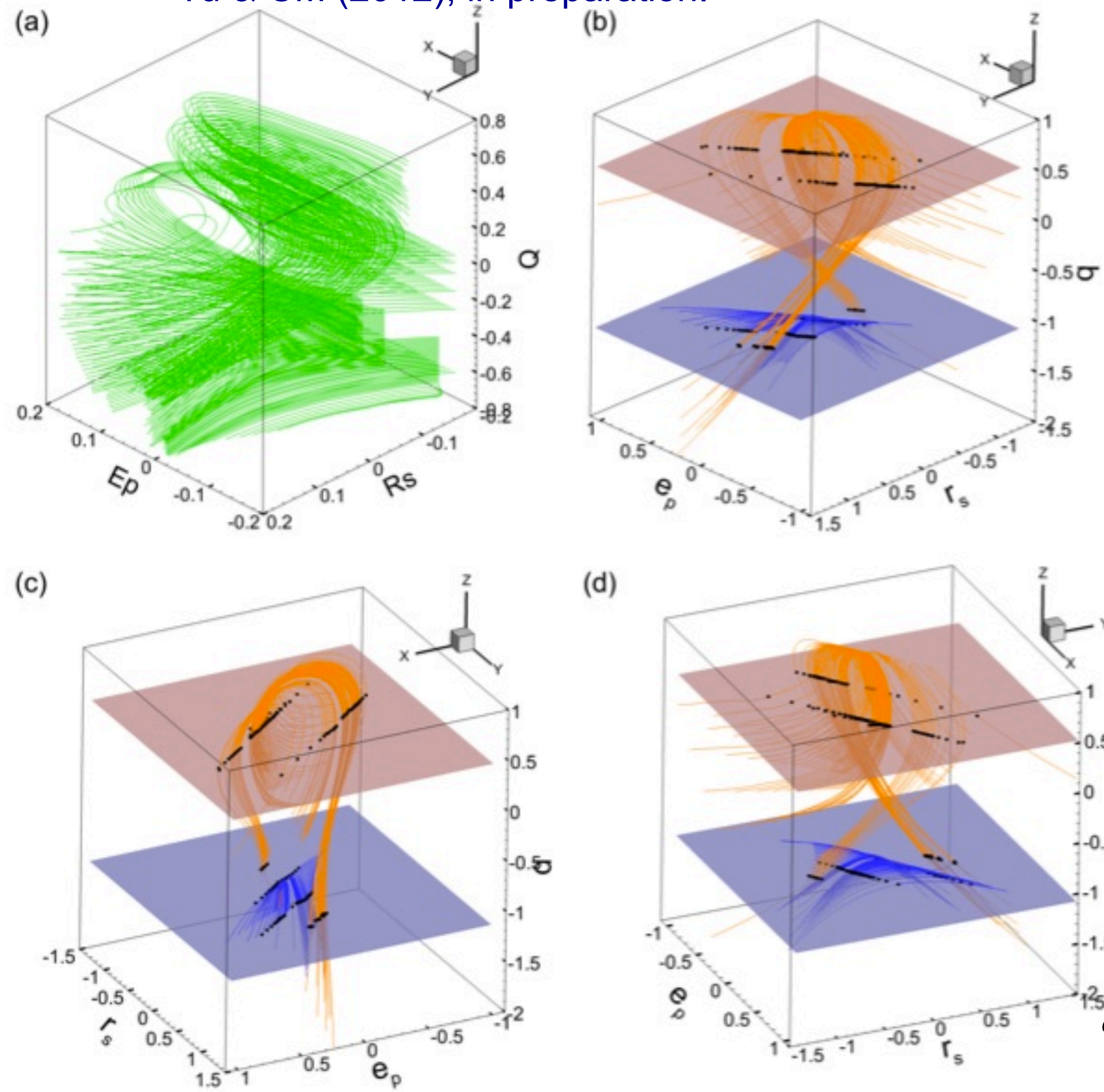
$$R_S = -\frac{1}{3}\mathrm{Tr}(\mathbf{S}^3)$$

$$E_P = \frac{1}{4}\omega_i\omega_j S_{ij}$$

$$R = R_S - E_P$$

Same, but in 3D expanded RQ-diagram:

Yu & CM (2012), in preparation.



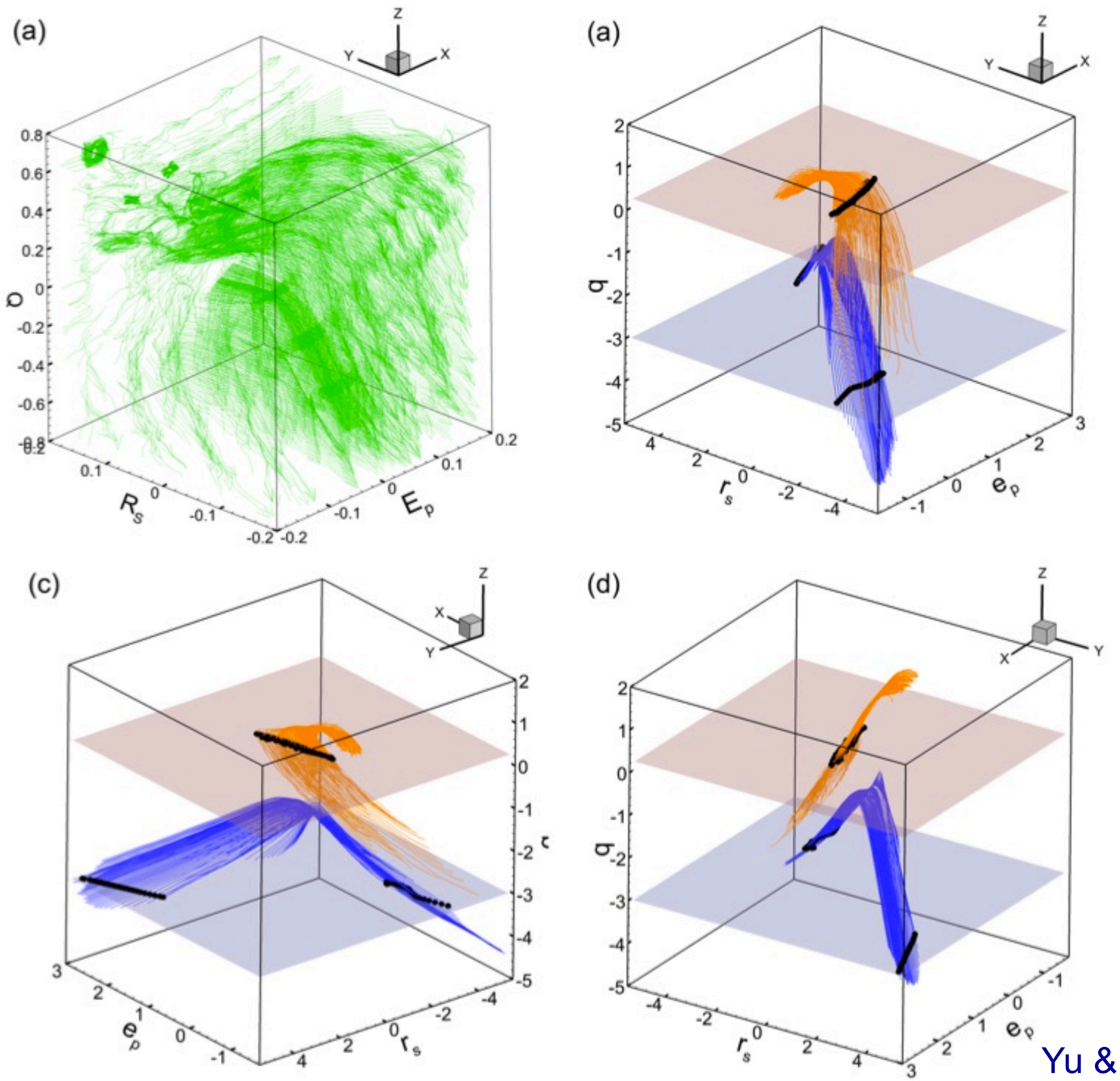
$$R_S = -\frac{1}{3}\mathrm{Tr}(\mathbf{S}^3)$$

$$E_P = \frac{1}{4}\omega_i\omega_j S_{ij}$$

$$R = R_S - E_P$$

$$q = \left(sign(Q_0) - \frac{27}{4}(r_s - e_p)^2 \right)^{3/2}$$

Pressure Hessian in 3D expanded RQ-diagram:



$$R_S = -\frac{1}{3}\mathrm{Tr}(\mathbf{S}^3)$$

$$E_P = \frac{1}{4}\omega_i\omega_j S_{ij}$$

$$R = R_S - E_P$$

$$q = \frac{Q}{Q_{P0}}$$

$$r_S = \frac{R_S}{Q_{PO}^{3/2}}$$

$$e_S = \frac{E_S}{Q_{PO}^{3/2}}$$

Yu & CM (2012), in preparation.

as promised, no conclusions

thanks