Some parabolic models for chemotaxis in 2D

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1. The Patlak, Keller & Segel model

The Keller & Segel model for chemotaxis consists of two coupled parabolic equations:
- an advection-diffusion equation for the evolution of cell density \( n(t,x) \),
- a reaction-diffusion equation for the evolution of chemical concentration \( c(t,x) \).

Several variants of the following system have been studied:

\[
\begin{align*}
\partial_t n + \nabla \cdot ( -\nabla n + \chi n \nabla c ) &= 0 \quad t \geq 0, \quad x \in \Omega \subset \mathbb{R}^2 \\
\Gamma \partial_t c - \Delta c &= n - \alpha c
\end{align*}
\]
Particularly the degenerate case under the assumption of high diffusion of chemical species [Jäger & Luckhaus]

$$\begin{align*}
\partial_t n + \nabla \cdot (-\nabla n + \chi n \nabla c) &= 0 \quad t \geq 0, \ x \in \Omega \\
-\Delta c &= n - \int n
\end{align*}$$

The first task is to study whether or not solutions of these coupled equations blow-up (in finite time).

The main result is the following.

**Theorem 1** *There exists a constant $C^*$ such that if $\chi M < C^*$ then the system admits global in time solution.*

At least two distinct approaches can be useful in order to prove this theorem.
1.1. *A priori* estimates

One can derive *a priori* estimates based on the following computation

\[
\frac{d}{dt} \int \Phi(n) dx = \int -\Phi''(n)|\nabla n|^2 dx + \chi \int n\psi(n)dx,
\]

with \(\psi'(x) = x\Phi''(x)\)

\(\Phi(x)\) is a convex function growing faster than \(x\) near infinity, typically \(\Phi(x) = x \ln x\).

It is possible to estimate the balance between the two terms, corresponding respectively to diffusion and aggregation of cells, thanks to a Gagliardo-Nirenberg-Sobolev inequality

\[
\int n^2 \leq C_{GNS} M \int |\nabla \sqrt{n}|^2.
\]

If dimension \(d = 2\), the total mass of cells \(M = \|n\|_{L^1}\) appears naturally from this inequality.
Consequently, the equi-integrability allows controls of the \( L^p \) norms of the cell density \( n \), by another computation with \( \Phi(x) = (x - k)_+^p \).

\[
\frac{d}{dt} \int_\Omega (n - k)^p_+ + C \left\{ 1 - C \int_\Omega (n - k)_+ \right\} \int_\Omega |\nabla (n - k)^{p/2}_+|^2 \\
\leq Ck \int_\Omega (n - k)^p_+ + Ck^2 \left( \int_\Omega (n - k)^p_+ \right)^{1-1/(p-1)}
\]
1.2. The energy of the system in the case of $\Omega$ bounded

There is an energy for the previous system

$$\begin{cases} \partial_t n = \nabla \cdot \left\{ n \nabla (\ln n - \chi c) \right\} & t \geq 0, \ x \in \Omega \subset \mathbb{R}^2, \\ -\Delta c = n - \int n, \end{cases}$$

which is of the following type

$$E(t) = \int n \ln n - \frac{\chi}{2} \int nc, \quad \frac{dE}{dt} = - \int n |\nabla (\ln n - \chi c)|^2 \leq 0.$$ 

Introducing the stationary states of the system, it is possible to show that $\int |\nabla c|^2$ remains bounded. As a consequence so does $\int n \ln n$. 
A Sobolev-type inequality is used in the critical case of the imbedding

$$H^1(\Omega) \hookrightarrow L_A(\Omega)$$

where $L_A$ is the Orlicz space associated to the convex function $A(s) = \exp(s^2)$.

**Lemma 1 (Trudinger & Moser)** If $u \in H^1(\Omega)$ and $\int u = 0$ (Neumann Boundary Conditions) then

$$\int e^u \leq C \exp\left(\frac{1}{8\pi} \int |\nabla u|^2\right)$$
2. Model for angiogenesis

Another very studied model for cell movement is angiogenesis. In its simplest form, the system is

\[
\begin{align*}
\partial_t n &= \Delta n - \nabla \cdot (n \chi(c) \nabla c) \quad t \geq 0 \, , \, x \in \mathbb{R}^2 \\
\partial_t c &= -nc
\end{align*}
\]

This system also admits an energy, given by

\[
\mathcal{E}(t) = \int n \ln n + \frac{1}{2} \int |\nabla \Phi(c)|^2, \quad \frac{d\mathcal{E}}{dt} \leq 0
\]

provided \( \inf_{c \geq 0} \left\{ \frac{cx'}{\chi} + 1 \right\} \geq 0 \); where \( \Phi \) is defined by the differential equation

\[
\Phi'(c) = \sqrt{\frac{\chi(c)}{c}}.
\]

This estimation reveals that the family \( \{n(t) \ln n(t)\} \) is equi-integrable.
Nevertheless, in order to control the $L^p$ norms of $n$ as in the previous section, another strategy has to be stated. For instance, it is possible to transform the first equation into a divergence form

$$\partial_t \left( \frac{n}{\phi(c)} \right) = \frac{1}{\phi(c)} \nabla \cdot \left\{ \phi(c) \nabla \left( \frac{n}{\phi(c)} \right) \right\} + \left( \frac{n}{\phi(c)} \right)^2 \phi(c) \chi(c)c,$$

where $\phi(c)$ is defined by another differential equation

$$\phi'(c) = \phi(c) \chi(c).$$

It is then possible to reproduce and adapt computations of $\frac{d}{dt} \int f \left( \frac{n}{\phi(c)} \right) \phi(c)$ and to apply similarly the Gagliardo-Nirenberg-Sobolev inequalities in the case of $f(x) = (x - k)_+^p$. 
3. The generalized Keller & Segel model

To further studying different chemotactic models, we have chosen a generalization of the Keller & Segel model. It has been proposed by Tyson & Murray for the modelisation of spatial organisation in bacterial population.

\[
\begin{align*}
\frac{\partial_t n}{t \geq 0, x \in \mathbb{R}^2} &= \Delta n - \nabla \cdot (n \chi \nabla c) \\
-\Delta c &= nf \\
\frac{\partial_t f}{t \geq 0, x \in \mathbb{R}^2} &= -nf
\end{align*}
\]

Assumption of an additional chemical species: the stimulant \( f \) is necessary to produce the chemoattractant \( c \).
And \( f \) is only consumed by the cells.
It renders an account of short and long range effects because of the diffusion of the chemical \( c \), contrary to the local effect of the stimulant \( f \).
Unfortunately, we know no energy structure for this system of three coupled equations, which makes it dramatically different from the previous ones.

We present here a first draft to understand the behavior of this system.

Indeed, if we simply reproduce the first method presented above, based on the *a priori* estimation

\[
\frac{d}{dt} \int n \ln n = -4 \int |\nabla \sqrt{n}|^2 + \chi \|f\|_\infty \int n^2,
\]

we can’t hope gaining anything but the condition \(\chi \|f\|_\infty M < C^*\).

This condition is not satisfying: it doesn’t bring anything new by comparison to the classical Keller & Segel model; and it doesn’t capture the feature of the additional equation \(\partial_t f = -nf\).
Another approach consists of finding a combinaison of the following type

\[ W(t) = \int n \ln n + \beta \int nf^\gamma + \alpha \frac{1}{2} \int |\nabla f^\delta|^2, \]

which is decreasing for well-chosen values of \( \alpha \) and \( \beta \), and under some conditions involving \( \chi \| f \|_\infty \) and \( M \). We first compute \( \frac{d}{dt} W \)

\[
\frac{d}{dt} \int n \ln n = -4 \int |\nabla \sqrt{n}|^2 + \chi \int n^2 f, \\
\beta \frac{d}{dt} \int nf^\gamma = -\beta \int \nabla n \cdot \nabla f^\gamma + \chi \beta \int n \nabla c \cdot \nabla f^\gamma - \gamma \beta \int n^2 f^\gamma, \\
\alpha \frac{d}{dt} \frac{1}{2} \int |\nabla f^\delta|^2 = -\frac{\delta}{2} \alpha \int \nabla n \cdot \nabla f^{2\delta} - \delta \alpha \int n|\nabla f^\delta|^2. 
\]
In order to eliminate the bad contribution of the no-sign terms and the positive one, we’ll associate them with negative ones in two ways. The first group includes

\[-4 \int |\nabla \sqrt{n}|^2 - \left\{ \frac{\beta}{\delta} \int \nabla n \cdot \nabla f^\gamma \right\} - \delta \alpha \int n|\nabla f^\delta|^2,\]

and the second one includes

\[\chi \int n^2 f - \gamma \beta \int n^2 f^\gamma.\]

The unfriendly term $\chi \beta \int n \nabla c \cdot \nabla f^\gamma$ plays an ambivalent role in this description.
3.1. The first association

We force a remarkable square to appear thanks to the extrem terms. One can easily be convinced that we have to set $\delta \leq \gamma$. Under this assumption, we are able to dominate

$$-4 \int |\nabla \sqrt{n}|^2 + 2\beta \frac{\gamma}{\delta} \|f\|_{\gamma-\delta} \int |\nabla \sqrt{n}| \cdot |\sqrt{n} \nabla f^\delta| - \delta \alpha \int n|\nabla f^\delta|^2.$$ 

A first condition appears for the homogeneity of $\alpha$ and $\beta$, for this expression to be non-positive.

$$\left(\beta \|f\|_{\gamma-\delta} \frac{\gamma}{\delta}\right)^2 \equiv \alpha \delta.$$ 

The same computation arises for the other term of the same type $-\frac{\delta}{2} \alpha \int \nabla n \cdot \nabla f^{2\delta}$ and we get another homogeneity condition

$$\delta \alpha \|f\|_{\infty}^{2\delta} \equiv 1.$$
3.2. What about $\int n\nabla c \cdot \nabla f^\gamma$?

We can combine this no-sign term in a general way

$$\int n|\nabla c \cdot \nabla f^\gamma| = \frac{\gamma}{\gamma - \xi} \int n f^\xi |\nabla c \cdot \nabla f^{\gamma - \xi}|$$

$$\leq \left(\frac{\gamma}{\gamma - \xi}\right)^2 \frac{K}{2} \int n|\nabla f^{\gamma - \xi}|^2 + \frac{1}{2K} \int n f^{2\xi} |\nabla c|^2,$$

with a homogeneity constant $K$ which has to be determined.

We associate the first r.h.s term with $-\delta \alpha \int n|\nabla f^\delta|^2$. We set $\delta \leq \gamma - \xi$ for this purpose.

It follows

$$\chi K \frac{\beta}{2} \left(\frac{\gamma}{\delta}\right)^2 \| f \|_{\infty}^{2(\gamma - \xi - \delta)} \int n|\nabla f^\delta|^2 - \alpha \delta \int n|\nabla f^\delta|^2,$$

and we get an additional homogeneity condition for $K$

$$\chi \beta \| f \|_{\infty}^{2(\gamma - \xi - \delta)} \left(\frac{\gamma}{\delta}\right)^2 K \equiv \alpha \delta.$$
The second r.h.s term will be eliminated thanks to a combination of Sobolev and Gagliardo-Nirenberg-Sobolev inequalities

\[ \|\nabla c\|_4^4 \leq C_S \|nf\|_{4/3}^4, \]

\[ \left( \int n^{4/3} \right)^3 \leq C_{GNS} M^3 \int |\nabla \sqrt{n}|^2. \]

So that

\[ \int nf^{2\xi} |\nabla c|^2 \leq \frac{L}{2} \int n^2 f^\omega + \frac{1}{2L} \int f^\theta |\nabla c|^4, \]

with the relation \( \omega + \theta = 4\xi \), and also

\[ \int f^\theta |\nabla c|^4 \leq C^* \|f\|_{4+\theta} M^3 \int |\nabla \sqrt{n}|^2. \]

Finally we have to deal with the last remaining terms, namely \( \int n^2 f^\omega \) and \( \int n^2 f \).
3.3. The second association

In order to eliminate those two terms, we of course associate them with $\int n^2 f^\gamma$. Only the case $\gamma \geq \max(1, \omega)$ is able to keep the homogeneity of the computations.

We use the following majoration which makes the distinction between high and low values of $f$

$$X^\omega \leq \kappa^{\omega} c_\nu + \kappa^{\gamma-\omega} X^\gamma , \quad \omega \nu = \gamma,$$

with the constant $E_\nu = c_\nu^{\nu-1} = \frac{\nu!}{\nu^{\nu}}$.

For each term $\int n^2 f^\omega$ and $\int n^2 f$ we get two new terms involving $\int n^2$ and $\int n^2 f^\gamma$.

We can use the first cited G.N.S. inequality to estimate $\int n^2$ : we have determinated all the homogeneity constants introduced

$$\chi \beta K^{-1} L^{-1} \|f\|_\infty^{\theta+4} M^3 \equiv 1,$$

$$\chi (c_\gamma \chi M)^{\gamma-1} \equiv \gamma \beta,$$

$$\chi \beta K^{-1} L (\chi \beta K^{-1} L c_\nu M)^{\nu-1} \equiv \gamma \beta.$$
3.4. Consequences of the homogeneity relations and conclusion

Using these six homogeneity conditions, we can eliminate all the intermediate parameters, and finally we get two different consequences of these relations

\[ E_{\gamma} \chi^\gamma \|f\|_\infty M^{\gamma^{-1}} \equiv \delta, \]

and

\[ E_{\nu} \chi^{4\nu} \|f\|_{4\nu} M^{4\nu^{-1}} \equiv \delta. \]

Consequently we assume \( \gamma = 4\nu \) to unify these two relations, which forces \( \gamma \geq 4 \) and \( \delta \leq \gamma - \xi \leq \gamma - \frac{1}{4}(\omega + \theta) \leq \gamma - 1. \)

