Nonlocal pressure and viscous contributions to the velocity gradient statistics based on Gaussian random fields

Michael Wilczek[1] & Charles Meneveau[2]

[1]MPI for Dynamics and Self-Organization, Göttingen, Germany
[2]Department of Mechanical Engineering, Johns Hopkins University

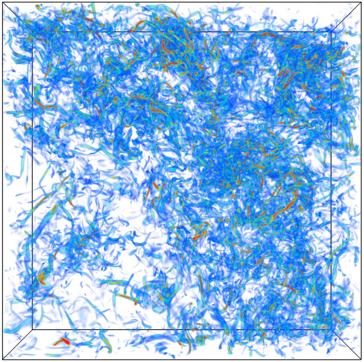
WPI Workshop on "Basic issues of extreme events in turbulence"
Vienna, May 4th-8th 2015

Overview

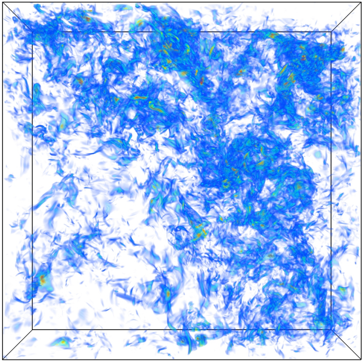
- small-scale features of homogeneous isotropic turbulence
- statistical description: closure problem
- closure based on Gaussian random fields
- comparison to DNS data

Small-Scale Structures in Turbulence

vorticity field $\omega^2/2$



strain field $\mathrm{Tr}(\mathrm{S}^2)$



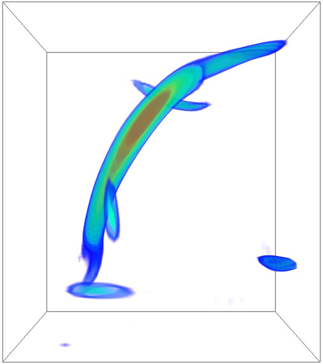
homogeneous isotropic turbulence exhibits intermittent distribution of

- vortex filaments
- strain sheets

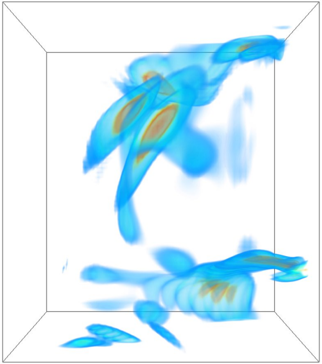
goal: derive dynamical model to capture essential statistical features

Small-Scale Structures in Turbulence

vorticity field $\boldsymbol{\omega}^2/2$



strain field $\mathrm{Tr}(\mathrm{S}^2)$



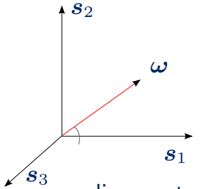
homogeneous isotropic turbulence exhibits intermittent distribution of

- ▷ vortex filaments
- ▷ strain sheets

goal: derive dynamical model to capture essential statistical features

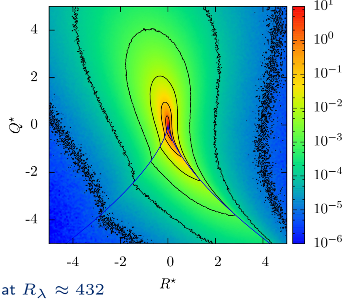
Velocity Gradient Tensor Statistics

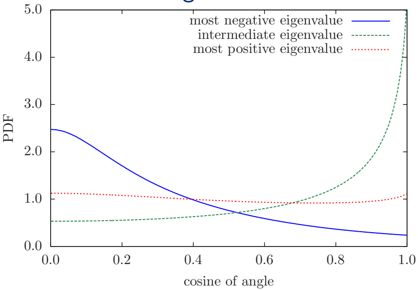
- velocity gradients $A_{ij} = \frac{\partial u_i}{\partial x_j} = S_{ij} + W_{ij}$ contains rich information on:
  - vorticity
  - strain
  - geometry



alignment

$Q = -\frac{1}{2}\mathrm{TrA}^2 \quad R = -\frac{1}{3}\mathrm{TrA}^3$





DNS data at $R_\lambda \approx 432$

http://turbulence.pha.jhu.edu

Velocity Gradient Tensor Dynamics

$$\frac{\mathrm{D}}{\mathrm{D}t}\mathrm{A}(\boldsymbol{x},t) = -\mathrm{A}(\boldsymbol{x},t)^2 - \mathrm{H}(\boldsymbol{x},t) + \nu\Delta\mathrm{A}(\boldsymbol{x},t) + \mathrm{F}(\boldsymbol{x},t)$$

with pressure Hessian:

$$\mathrm{H} = \underbrace{-\frac{1}{3}\mathrm{Tr}\left(\mathrm{A}^2\right)\mathrm{I}}_{\text{local (isotropic) part.}} + \underbrace{\widetilde{\mathrm{H}}}_{\text{nonlocal part}}$$

$$\widetilde{H}_{ij}(\boldsymbol{x},t) = -\frac{1}{4\pi}\int_{\mathrm{PV}}\mathrm{d}\boldsymbol{x}'\left[\frac{\delta_{ij}}{|\boldsymbol{x}'-\boldsymbol{x}|^3} - 3\frac{(\boldsymbol{x}'-\boldsymbol{x})_i(\boldsymbol{x}'-\boldsymbol{x})_j}{|\boldsymbol{x}'-\boldsymbol{x}|^5}\right]\mathrm{Tr}\left(\mathrm{A}(\boldsymbol{x}',t)^2\right)$$

- ▷ closure problem/modeling challenge for statistical description: express $\widetilde{\mathrm{H}}$ and $\nu\Delta\mathrm{A}$ as function of A

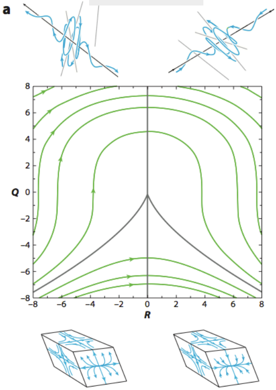
[Ohkitani & Kishiba, Phys. Fluids 7, 411 (1995)]

[see also: Meneveau, Annu. Rev. Fluid Mech. 43, 245 (2011)]

Restricted Euler & Linear Diffusion Models

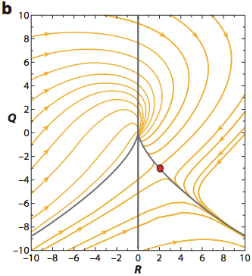
Restricted Euler approximation

$$\widetilde{\mathrm{H}} = 0$$
$$\nu \Delta \mathrm{A} = 0$$



Linear diffusion model

$$\widetilde{\mathrm{H}} = 0$$
$$\nu \Delta \mathrm{A} = -\frac{\mathrm{A}}{\tau_0}$$



[Vieillefosse, J. Phys. 43, 837 (1982)]

[Cantwell, Phys. Fluids. A 4, 782 (1992)]

[Martin et al., Phys. Fluids 10, 2012 (1998)]

figs. from [Meneveau, Annu. Rev. Fluid Mech. 43, 245 (2011)]

advanced models: [Chertkov et. al., Phys. Fluids 11, 2394 (1999)], [Naso & Pumir, PRE 72, 056318 (2005)], [Chevillard & Meneveau, PRL 97, 174501 (2006)]

Statistical Evolution Equation & Gaussian Random Field Closure

- from PDF equation: exact (but unclosed!) statistical evolution equation:

$$\mathrm{d}\mathcal{A} = \left[-\left(\mathcal{A}^2 - \frac{1}{3}\mathrm{Tr}(\mathcal{A}^2)\mathrm{I} \right) - \langle \widetilde{\mathrm{H}} | \mathcal{A} \rangle + \langle \nu \Delta \mathrm{A} | \mathcal{A} \rangle \right] \mathrm{d}t + \mathrm{dF}$$

- nonlocal pressure Hessian:

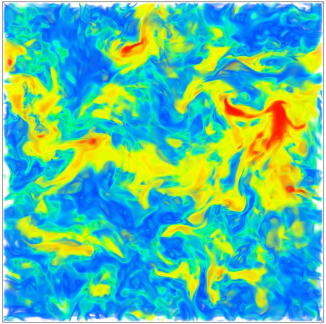
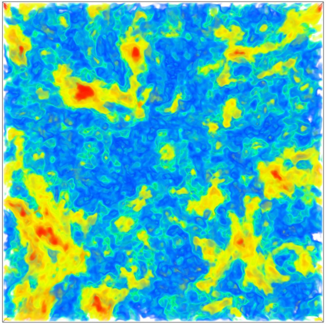
$$\langle \widetilde{H}_{ij}(\boldsymbol{x}_1) | \mathcal{A}_1 \rangle = -\frac{1}{4\pi} \int_{\mathrm{PV}} \mathrm{d}\boldsymbol{r} \left[\frac{\delta_{ij}}{r^3} - 3\frac{r_i\, r_j}{r^5} \right] \langle \mathrm{Tr}(A^2)(\boldsymbol{x}_1 + \boldsymbol{r}) | \mathcal{A}_1 \rangle$$

- viscous term:

$$\langle \nu \Delta_{\boldsymbol{x}_1} \mathrm{A}(\boldsymbol{x}_1, t) | \mathcal{A}_1 \rangle = \lim_{r \to 0} \nu \Delta_{\boldsymbol{r}} \langle \mathrm{A}(\boldsymbol{x}_1 + \boldsymbol{r}, t) | \mathcal{A}_1 \rangle$$

- Closure needs specification of a random field!

Idea: Gaussian Random Field Closure

"replace  with  "

Wilczek & Meneveau, J. Fluid Mech. 756, 191 (2014)

Incompressible Gaussian Velocity Fields

- def.: every finite-dimensional density is multivariate Gaussian
- comprehensive statistical description: characteristic functional

$$\begin{aligned}\phi^u[\boldsymbol{\lambda}(\boldsymbol{x})] &= \left\langle \exp\left[\mathrm{i}\int \mathrm{d}\boldsymbol{x}\,\lambda_i(\boldsymbol{x})\,u_i(\boldsymbol{x})\right]\right\rangle \\ &= \exp\left[-\frac{1}{2}\int \mathrm{d}\boldsymbol{x}\int \mathrm{d}\boldsymbol{x}'\,\lambda_i(\boldsymbol{x})R^u_{ij}(\boldsymbol{x},\boldsymbol{x}')\lambda_j(\boldsymbol{x}')\right]\end{aligned}$$

- $R^u_{ij}(\boldsymbol{x},\boldsymbol{x}')$ is the velocity covariance tensor,
  specified by longitudinal autocorrelation function $f_u(r)$

Analytical Calculation: Details

1. start from characteristic functional for velocity ϕ^u
2. obtain characteristic functional for velocity gradient ϕ^{A}
3. calculate conditional averages:

$$\left\langle \mathrm{A}(\boldsymbol{x}_2) \middle| \mathcal{A}_1 \right\rangle \quad \text{and} \quad \left\langle \mathrm{Tr}\left(\mathrm{A}(\boldsymbol{x}_2)^2\right) \middle| \mathcal{A}_1 \right\rangle$$

4. evaluate conditional pressure Hessian and viscous term:

$$\left\langle \widetilde{H}_{ij}(\boldsymbol{x}_1) \middle| \mathcal{A}_1 \right\rangle \quad \text{and} \quad \left\langle \nu \Delta_{\boldsymbol{x}_1} \mathrm{A}(\boldsymbol{x}_1, t) \middle| \mathcal{A}_1 \right\rangle$$

Wilczek & Meneveau, J. Fluid Mech. 756, 191 (2014)

Gaussian Nonlocal Pressure Hessian Contributions

Gaussian closure:

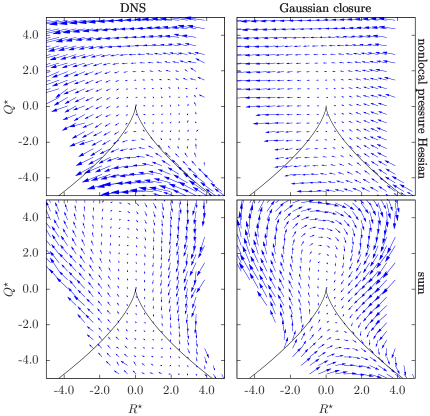
$$\left\langle \widetilde{\mathrm{H}}(\boldsymbol{x}_1) \middle| \mathcal{A}_1 \right\rangle = \alpha \left(\mathcal{S}_1^2 - \frac{1}{3} \mathrm{Tr}\left(\mathcal{S}_1^2\right) \mathrm{I} \right) + \beta \left(\mathcal{W}_1^2 - \frac{1}{3} \mathrm{Tr}\left(\mathcal{W}_1^2\right) \mathrm{I} \right) + \gamma \left(\mathcal{S}_1 \mathcal{W}_1 - \mathcal{W}_1 \mathcal{S}_1 \right)$$

$$\alpha = -\frac{2}{7} \approx -0.29$$

$$\beta = -\frac{2}{5} = -0.4$$

$$\gamma = \frac{6}{25} + \frac{16}{75 f_u''(0)^2} \int \mathrm{d}r \, \frac{f_u' f_u'''}{r} \approx 0.08$$

- quadratic expression of velocity gradient
- symmetric
- traceless



Gaussian Nonlocal Pressure Hessian Contributions

Gaussian closure:

$$
\begin{aligned}
\langle \widetilde{\mathrm{H}}(\boldsymbol{x}_1) | \mathcal{A}_1 \rangle = \alpha & \left(\mathcal{S}_1^2 - \frac{1}{3} \mathrm{Tr} \left(\mathcal{S}_1^2 \right) \mathrm{I} \right) \\
& + \beta \left(\mathcal{W}_1^2 - \frac{1}{3} \mathrm{Tr} \left(\mathcal{W}_1^2 \right) \mathrm{I} \right) \\
& + \gamma \left(\mathcal{S}_1 \mathcal{W}_1 - \mathcal{W}_1 \mathcal{S}_1 \right)
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= -\frac{2}{7} \approx -0.29 \\
\beta &= -\frac{2}{5} = -0.4 \\
\gamma &= \frac{6}{25} + \frac{16}{75 f_u''(0)^2} \int \mathrm{d}r \, \frac{f_u' f_u'''}{r} \approx 0.08
\end{aligned}
$$

- quadratic expression of velocity gradient
- symmetric
- traceless

Gaussian closure blows up!
generalization:

- improve coefficients based on DNS data
- enhanced Gaussian closure:

$$
\begin{aligned}
\alpha &= -0.61 \\
\beta &= -0.65 \\
\gamma &= 0.14
\end{aligned}
$$

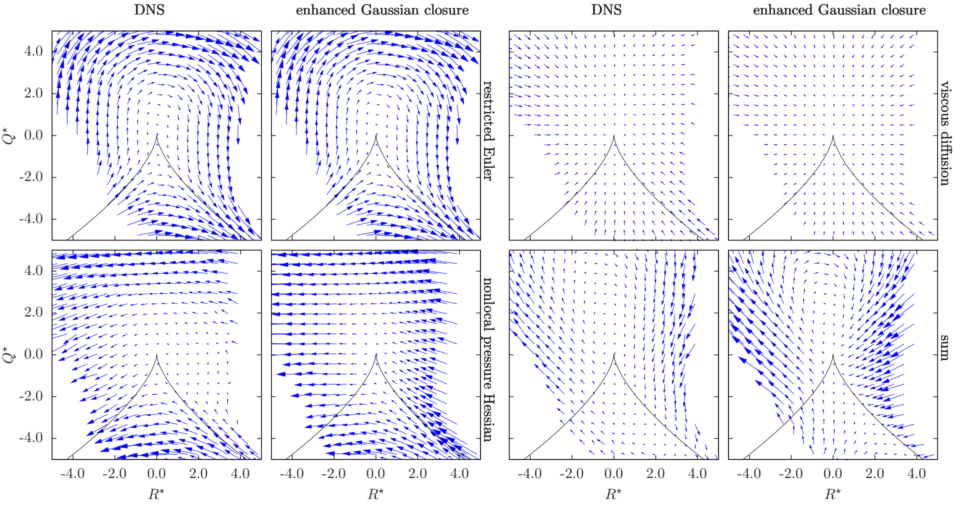
Gaussian Viscous Contribution

$$
\begin{aligned}
\left\langle \nu \Delta_{\boldsymbol{x}_1} \mathrm{A}(\boldsymbol{x}_1) \middle| \mathcal{A}_1 \right\rangle &= \delta \mathcal{A}_1 \\
\delta = \nu \frac{7}{3} \frac{f_u^{(4)}(0)}{f_u''(0)} &= -\nu \frac{\int \mathrm{d}k\, k^4\, E(k)}{\int \mathrm{d}k\, k^2\, E(k)}
\end{aligned}
$$

- Reynolds number dependence through autocorrelation function/spectrum
- Gaussian assumption consistent with linear diffusion models
- but additionally: coefficient fixed
- estimate from DNS:

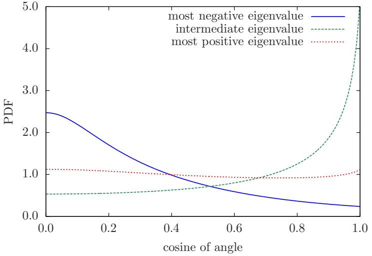
$$
\delta \tau_\eta = -0.15
$$

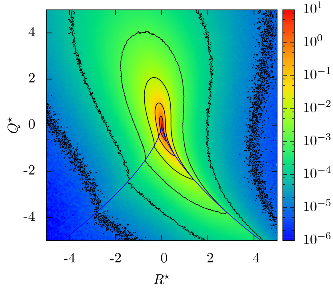
Comparison to DNS Data (a priori)



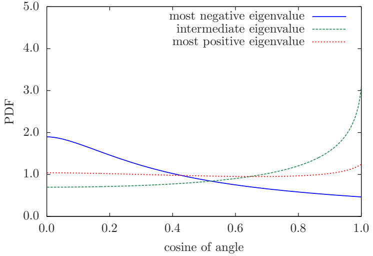
Comparison to DNS Data (stochastic ODE model)

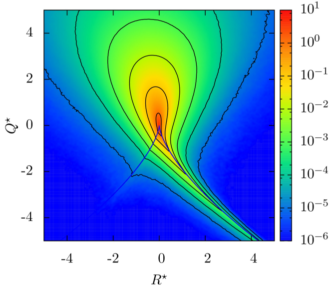
DNS



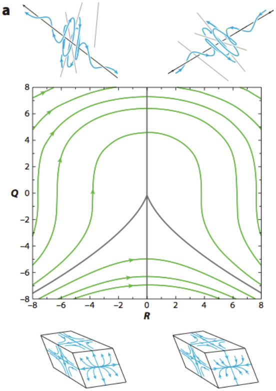


closure





Phenomenological view on the occurrence and decay of extreme events



$$\frac{\mathrm{D}}{\mathrm{D}t}\mathrm{A} = -\mathrm{A}^2 - \mathrm{H} + \nu\Delta\mathrm{A} + \mathrm{F}$$

- Restricted Euler nonlinearity produces steep gradients (structures?)
- nonlocal pressure Hessian builds up "restoring force" $\sim$ volume-weighted balance between strain and vorticity $(\mathrm{Tr}(\mathrm{A}^2) = \mathrm{Tr}(\mathrm{S}^2) + \mathrm{Tr}(\mathrm{W}^2))$
- viscous term damps structures $\sim$ local curvature

Conclusions

- pressure Hessian and viscous terms evaluated for Gaussian velocity fields
- Gaussian closure:
  - viscous term: linear damping
  - nonlocal pressure Hessian: combination of quadratic, traceless and symmetric velocity gradient expressions
- enhanced Gaussian closure:
  - adjusted coefficients to counterbalance restricted Euler singularity
- enhanced Gaussian closure leads to stable ODE model

Future Work and Open Questions

- How to construct non-Gaussian random fields for better closures?
- Use simple ODE models to predict structures/extreme events in turbulent flows?

Thank you!
Questions?